# Tethered Diamonds

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## 1 Syntax

$$
\begin{array}{rlll}
\text{Worlds} & p, q & ::= & \alpha \mid \cdots \\
\text{Right Judgments} & J & ::= & A \mid (A\,\mathrm{poss}^{\geq q}) \mid (A\,\mathrm{rdy}^{\geq q}) \\
\text{Contexts} & \Gamma & ::= & \cdot \mid \Gamma, A[p] \mid \Gamma, A[\geq p] \mid \Gamma, p \geq q \\
\text{Propositions} & A & ::= & \Box A \mid \Diamond A \mid A \multimap A \mid a \mid \cdots
\end{array}
$$

The general form of the sequent is $\Gamma \vdash J[p]$. The judgment rdy stands for 'ready'. It is an intermediate stage of the possibility judgment where one left rule has been applied and it is licensed to go back to truth even in the absence of 'monadic reflexivity' (rule $jR[T]$). But it is more useful than plain truth: in the presence of 'monadic transitivity' (rule $\Diamond L[4]$) it can cycle back to rdy under a further use of the left rule.

## 2 Rules

First some boring rules to establish what tethering means:

$$
\frac{}{\Gamma, a[p] \vdash a[p]}\,init \qquad \frac{}{\Gamma, p \geq q \vdash p \geq q}\,init \geq
$$

$$
\frac{\Gamma, A[p] \vdash B[p]}{\Gamma \vdash A \multimap B[p]} \multimap R \qquad \frac{\Gamma \vdash A[p] \qquad \Delta, B[p] \vdash J[p]}{\Gamma, \Delta, A \multimap B[p] \vdash J[p]} \multimap L
$$

Here's the rules for $\Box$, and for the 'validity' judgment on the left

$$
\frac{\Gamma, \alpha \geq p \vdash A[\alpha]}{\Gamma \vdash \Box A[p]} \Box R \qquad \frac{\Gamma, A[\geq p] \vdash J[p]}{\Gamma, \Box A[p] \vdash J[p]} \Box L
$$

$$
\frac{\Gamma \vdash p \geq q \qquad \Gamma, A[p] \vdash J[p]}{\Gamma, A[\geq q] \vdash J[p]}\, jL
$$

Finally here are the rules for $\Diamond$.

$$\frac{\Gamma \vdash (A\,\mathrm{poss}^{\geq p})[p]}{\Gamma \vdash \Diamond A[p]}\ \Diamond R$$

$$\frac{p \geq q \qquad \Gamma \vdash A[p]}{\Gamma \vdash (A\,\mathrm{rdy}^{\geq q})[p]}\ jR \qquad \frac{p \geq q \qquad \Gamma \vdash A[p]}{\Gamma \vdash (A\,\mathrm{poss}^{\geq q})[p]}\ jR[T]$$

$$\frac{\Gamma, \alpha \geq p, A[\alpha] \vdash (C\,\mathrm{rdy}^{\geq q})[\alpha]}{\Gamma, \Diamond A[p] \vdash (C\,\mathrm{poss}^{\geq q})[p]}\ \Diamond L \qquad \frac{\Gamma, \alpha \geq p, A[\alpha] \vdash (C\,\mathrm{rdy}^{\geq q})[\alpha]}{\Gamma, \Diamond A[p] \vdash (C\,\mathrm{rdy}^{\geq q})[p]}\ \Diamond L[4]$$

Rules $\Diamond R$, $\Diamond L$, and $jR$ are always included. $jR[T]$ and $\Diamond L[4]$ are independently optional; including them affects the *monadic* component of the $\Diamond$ in ways analogous to the $T$ and 4 axioms. Specifically I conjecture that if you appropriately axiomatize $\geq$ and choose the right set of rules, you get the modal logics in the weather report, like so:

| Weather Report | $\geq$ | $jR[T]$ | $\Diamond L[4]$ |
|---|---|---|---|
| K | no axioms | | |
| T | refl | ✓ | |
| 4 | trans | | ✓ |
| S4 | refl, trans | ✓ | ✓ |

But if one considers these four options for the Kripke relation, and these four options for the monadic behavior, there's actually a 4 by 4 grid of possibilites. For instance, we can axiomatize $\leq$ with no axioms and include both $jR[T]$ and $\Diamond L[4]$. This gives a logic in which $\Diamond\Diamond\bot \vdash \Diamond\bot$ but not $\Diamond\Diamond A \vdash \Diamond A$. At least this logic is distinct from the other four above, because with $\Diamond L[4]$ and transitivity of $\geq$, we can prove $\Diamond\Diamond A \vdash \Diamond A$, and without $\Diamond L[4]$, we cannot prove $\Diamond\Diamond\bot \vdash \Diamond\bot$. I would not tend to guess that all 16 possibilities are distinct.

## 2.1 Omitting the Kripke Mechanism

If we get rid of all mention of explicit worlds, we have four variants of $\bigcirc$:

$$\frac{\Gamma \vdash A\,\mathrm{poss}}{\Gamma \vdash \bigcirc A}\ \bigcirc R$$

$$\frac{\Gamma \vdash A}{\Gamma \vdash A\,\mathrm{rdy}}\ jR \qquad \frac{\Gamma \vdash A}{\Gamma \vdash A\,\mathrm{poss}}\ jR[T]$$

$$\frac{\Gamma, A \vdash C\,\mathrm{rdy}}{\Gamma, \bigcirc A \vdash C\,\mathrm{poss}}\ \bigcirc L \qquad \frac{\Gamma, A \vdash C\,\mathrm{rdy}}{\Gamma, \bigcirc A \vdash C\,\mathrm{rdy}}\ \bigcirc L[4]$$

Again, $jR[T]$ and $\bigcirc L[4]$ are independently optional. The familiar lax logic is the one we get by including both $jR[T]$ and $\bigcirc L[4]$.

I was hoping that this $\bigcirc$ with just $T$ and not 4 might be proof irrelevance, but proof irrelevance would prove $\bigcirc A \wedge \bigcirc B \vdash \bigcirc(A \wedge B)$ whereas this system does not.

## 2.2 Omitting the Monadic Mechanism

If we get rid of all the monadic tethering, what we get is another $\Diamond$-like operator parametrized over a Kripke relation: Instead of

$$\text{Right Judgments} \quad J ::= \; A \mid (A\,\mathrm{poss}^{\geq q}) \mid (A\,\mathrm{rdy}^{\geq q})$$

we have simply

$$\text{Right Judgments} \quad J ::= \; A \mid (A \geq q)$$

and rules

$$\frac{\Gamma \vdash (A \geq p)[p]}{\Gamma \vdash \Diamond A[p]}\,\Diamond R$$

$$\frac{p \geq q \qquad \Gamma \vdash A[p]}{\Gamma \vdash (A \geq q)[p]}\,jR$$

$$\frac{\Gamma, \alpha \geq p, A[\alpha] \vdash J[\alpha]}{\Gamma, \Diamond A[p] \vdash J[p]}\,\Diamond L$$

These systems all prove $\Diamond\bot \vdash \bot$ regardless of $\leq$, yet do not prove $\Diamond(A \vee B) \vdash \Diamond A \vee \Diamond B$.