Canonical Forms in a Hybrid Logical Framework

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1 Language

$$
\begin{array}{rrcl}
\text{Worlds} & p, q, r & ::= & \alpha \mid p \cdot q \mid \epsilon \\
\text{Kinds} & K & ::= & \Pi x{:}A.K \mid \forall\alpha.K \mid \text{type} \\
\text{Types} & A & ::= & \Pi x{:}A.B \mid a \cdot S \mid \forall\alpha.B \mid {\downarrow}\alpha.B \mid @_p A \mid A \mathbin{\&} B \mid \top \\
\text{Terms} & M & ::= & \lambda x.M \mid R \mid \langle M_1, M_2\rangle \mid \langle\rangle \\
\text{Atomic Terms} & R & ::= & c \cdot S \mid x \cdot S \\
\text{Spines} & S & ::= & () \mid (M; S) \mid (\pi_i; S)
\end{array}
$$

1.1 Judgments

$$
\begin{array}{l}
\Gamma \vdash K : \text{rkind} \\
\Gamma \vdash p : \mathsf{world} \\
\Gamma \vdash A : \text{type} \\
\Gamma \vdash S : K > \text{type} \\
\Gamma \vdash M : A[p] \\
\Gamma \vdash R : A[p] \\
\Gamma \vdash S : A[p] > C[r]
\end{array}
$$

1.2 Kind Formation

$$
\frac{\Gamma, x : A \vdash K : \text{rkind}}{\Gamma \vdash \Pi x : A.K : \text{rkind}} \qquad \frac{\Gamma, \alpha : \mathsf{world} \vdash K : \text{rkind}}{\Gamma \vdash \forall\alpha.K : \text{rkind}}
$$

$$
\frac{}{\Gamma \vdash \text{type} : \text{rkind}}
$$

1.3 World Formation

$$
\frac{\alpha : \mathsf{world} \in \Gamma}{\Gamma \vdash \alpha : \mathsf{world}} \qquad \frac{\Gamma \vdash p : \mathsf{world} \qquad \Gamma \vdash q : \mathsf{world}}{\Gamma \vdash p \cdot q : \mathsf{world}} \qquad \frac{}{\Gamma \vdash \epsilon : \mathsf{world}}
$$

The relation $\equiv_{ACU}$ holds between two world expressions if they are identical up to associativity and commutativity for $\cdot$, and unit laws for ϵ.

1.4 Type Formation

$$\frac{\Gamma, x : A \vdash B : \text{type}}{\Gamma \vdash \Pi x{:}A.B : \text{type}}$$

$$\frac{a : K \in \Sigma \qquad \Gamma \vdash S : K > \text{type}}{\Gamma \vdash a \cdot S : \text{type}}$$

$$\frac{\Gamma, \alpha : \mathsf{world} \vdash B : \text{type}}{\Gamma \vdash \forall \alpha.B : \text{type}}$$

$$\frac{\Gamma, \alpha : \mathsf{world} \vdash B : \text{type}}{\Gamma \vdash \downarrow \alpha.B : \text{type}}$$

$$\frac{\Gamma \vdash A : \text{type} \qquad \Gamma \vdash p : \mathsf{world}}{\Gamma \vdash @_p A : \text{type}}$$

$$\frac{\Gamma \vdash A : \text{type} \qquad \Gamma \vdash B : \text{type}}{\Gamma \vdash A \mathbin{\&} B : \text{type}}$$

$$\frac{}{\Gamma \vdash \top : \text{type}}$$

1.5 Type Checking

$$\frac{\Gamma \vdash R : a \cdot S[p] \qquad S =_\alpha S' \qquad p \equiv_{ACU} q}{\Gamma \vdash R : a \cdot S'[q]}$$

$$\frac{\Gamma, x : A \vdash M : B[p]}{\Gamma \vdash \lambda x.M : \Pi x{:}A.B[p]} \qquad \frac{\Gamma, \alpha : \mathsf{world} \vdash M : B[p]}{\Gamma \vdash M : \forall \alpha.B[p]}$$

$$\frac{\Gamma \vdash M : (\{p/\alpha\}^{\mathsf{world}} B)[p]}{\Gamma \vdash M : \downarrow \alpha.B[p]} \qquad \frac{\Gamma \vdash M : A[q]}{\Gamma \vdash M : @_q A[p]}$$

$$\frac{\Gamma \vdash M_1 : A_1[p] \qquad \Gamma \vdash M_2 : A_2[p]}{\Gamma \vdash \langle M_1, M_2 \rangle : A_1 \mathbin{\&} A_2[p]} \qquad \frac{}{\Gamma \vdash \langle\rangle : \top[p]}$$

1.6 Type Synthesis

$$\frac{c : A \in \Sigma \qquad \Gamma \vdash S : A[\epsilon] > C[r]}{\Gamma \vdash c \cdot S : C[r]}$$

$$\frac{x : A \in \Gamma \qquad \Gamma \vdash S : A[\epsilon] > C[r]}{\Gamma \vdash x \cdot S : C[r]}$$

1.7 Type Spine Kinding

$$\frac{}{\Gamma \vdash () : \mathrm{type} > \mathrm{type}}$$

$$\frac{\Gamma \vdash M : A[\epsilon] \qquad \Gamma \vdash S : \{M/x\}^A K > \mathrm{type}}{\Gamma \vdash (M; S) : \Pi x{:}A.K > \mathrm{type}}$$

$$\frac{\Gamma \vdash p : \mathsf{world} \qquad \Gamma \vdash S : \{p/\alpha\}^{\mathsf{world}} K > \mathrm{type}}{\Gamma \vdash S : \forall \alpha.K > \mathrm{type}}$$

1.8 Term Spine Typing

$$\frac{}{\Gamma \vdash () : a \cdot S[p] > a \cdot S[p]}$$

$$\frac{\Gamma \vdash M : A[\epsilon] \qquad \Gamma \vdash S : \{M/x\}^A B[p] > C[r]}{\Gamma \vdash (M; S) : \Pi x{:}A.B[p] > C[r]}$$

$$\frac{\Gamma \vdash q : \mathsf{world} \qquad \Gamma \vdash S : \{q/\alpha\}^{\mathsf{world}} B[p] > C[r]}{\Gamma \vdash S : \forall \alpha.B[p] > C[r]}$$

$$\frac{\Gamma \vdash S : \{p/\alpha\}^{\mathsf{world}} B[p] > C[r]}{\Gamma \vdash S : \downarrow\alpha.B[p] > C[r]}$$

$$\frac{\Gamma \vdash S : A[q] > C[r]}{\Gamma \vdash S : @_q A[p] > C[r]}$$

$$\frac{\Gamma \vdash S : A_i[p] > C[r]}{\Gamma \vdash (\pi_i; S) : A_1 \;\&\; A_2[p] > C[r]}$$

1.9 Substitution

Partial functions on normal terms yielding normal terms:

$$\{N/x\}^C \qquad \{q/\beta\}^{\mathsf{world}} \qquad [N \mid S]^B$$

The first two also takes kinds to kinds and types to types.

Let σ abbreviate $\{N/y\}^C$ or $\{q/\beta\}^{\mathsf{world}}$.

1.9.1 Substitution on Kinds

$$\sigma(\Pi x{:}A.K) = \Pi x{:}(\sigma A).(\sigma K)$$

$$\sigma(\forall \alpha.K) = \forall \alpha.(\sigma K)$$

$$\sigma \mathrm{type} = \mathrm{type}$$

1.9.2 Substitution on Types

$$\sigma(\Pi x{:}A.B) = \Pi x{:}(\sigma A).(\sigma B)$$

$$\sigma(a \cdot S) = a \cdot (\sigma S)$$

$$\sigma(\forall \alpha.B) = \forall \alpha.(\sigma B)$$

$$\sigma(\downarrow \alpha.B) = \downarrow \alpha.(\sigma B)$$

$$\sigma(@_p A) = @_{\sigma p}(\sigma A)$$

$$\sigma(A \ \& \ B) = \sigma A \ \& \ \sigma B$$

$$\sigma \top = \top$$

1.9.3 Substitution on Terms

$$\sigma(\lambda x.M) = \lambda x.(\sigma M)$$

$$\sigma\langle M_1, M_2\rangle = \langle \sigma M_1, \sigma M_2\rangle$$

$$\sigma\langle\rangle = \langle\rangle$$

$$\sigma\epsilon = \epsilon$$

$$\sigma(p \cdot q) = \sigma p \cdot \sigma q$$

$$\sigma(c \cdot S) = c \cdot (\sigma S)$$

$$\sigma(y \cdot S) = y \cdot (\sigma S) \qquad \text{(if } \sigma \text{ not for } y\text{)}$$

$$\{N/x\}^C(x \cdot S) = [N \mid S]^C$$

$$\sigma\beta = \beta \qquad \text{(if } \sigma \text{ not for } \beta\text{)}$$

$$\{p/\alpha\}^{\mathsf{world}}\alpha = p$$

$$\sigma() = ()$$

$$\sigma(M; S) = (\sigma M; \sigma S)$$

$$\sigma(\pi_i; S) = (\pi_i; \sigma S)$$

1.10 Reduction

$$[\lambda x.M \mid (N; S)]^{\Pi x:A.B} = [\{N/x\}^A M \mid S]^{\{M/x\}^A B}$$

$$[R \mid ()]^{a \cdot S} = R$$

$$[M \mid S]^{\forall \alpha.A} = [M \mid S]^A$$

$$[M \mid S]^{\downarrow \alpha.A} = [M \mid S]^A$$

$$[M \mid S]^{@_p A} = [M \mid S]^A$$

$$[\langle M_1, M_2\rangle \mid (\pi_i; S)]^{A_1 \& A_2} = [M_i \mid S]^{A_i}$$