Classical Truth as a Modal Operator

Jason Reed

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We have a judgment $\Gamma \vdash \gamma$ where contexts are defined by

$$\Gamma ::= \cdot \mid \Gamma, A \mid \Gamma, A\,\mathsf{t} \mid \Gamma, A\,\mathsf{f}$$

and propositions by

$$A, B, C ::= P \mid \bot \mid \mathbf{C}A \mid A \supset B$$

and the optionally empty conclusion γ is

$$\gamma ::= \cdot \mid A$$

Hypotheses $A\,\mathsf{t}$ and $A\,\mathsf{f}$ in Γ are assumptions of classical truth and falsehood. Bare hypotheses A in Γ are intuitionistic assumptions. The right-hand side of the judgment $\Gamma \vdash C$ is an intuitionistic conclusion. $\Gamma \vdash \cdot$ means that the assumptions in Γ are contradictory.

Propositions are either propositional variables P or falsehood $\bot$ or constructed from other propositions by the 'classically true' modality $\mathbf{C}$ and implication $\supset$.

The sequent calculus is as follows. The initial sequents are

$$\frac{}{\Gamma, P \vdash P}\,init_i \qquad \frac{}{\Gamma, P\,\mathsf{t}, P\,\mathsf{f} \vdash \gamma}\,init_c \qquad \frac{}{\Gamma, P, P\,\mathsf{f} \vdash \gamma}\,init_{ic}$$

The rules for $\bot$ are

$$\frac{}{\Gamma, \bot \vdash \gamma}\,\bot L$$

$$\frac{}{\Gamma, \bot\,\mathsf{t} \vdash \gamma}\,\bot T \qquad \frac{\Gamma \vdash \gamma}{\Gamma, \bot\,\mathsf{f} \vdash \gamma}\,\bot F$$

The rules for $\mathbf{C}$ are

$$\frac{\Gamma, A\,\mathsf{t} \vdash \gamma}{\Gamma, \mathbf{C}A \vdash \gamma}\,\mathbf{C}L \qquad \frac{\Gamma, A\,\mathsf{f} \vdash \cdot}{\Gamma \vdash \mathbf{C}A}\,\mathbf{C}R$$

$$\frac{\Gamma, A\,\mathsf{t} \vdash \gamma}{\Gamma, \mathbf{C}A\,\mathsf{t} \vdash \gamma}\,\mathbf{C}T \qquad \frac{\Gamma, A\,\mathsf{f} \vdash \gamma}{\Gamma, \mathbf{C}A\,\mathsf{f} \vdash \gamma}\,\mathbf{C}F$$

The rules for $\supset$ are

$$\frac{\Gamma, A \supset B \vdash A \qquad \Gamma, B \vdash \gamma}{\Gamma, A \supset B \vdash \gamma} \supset L \qquad \frac{\Gamma, A \vdash B}{\Gamma \vdash A \supset B} \supset R$$

$$\frac{\Gamma, A\,\mathsf{f} \vdash \gamma_1 \qquad \Gamma, B\,\mathsf{t} \vdash \gamma_2}{\Gamma, A \supset B\,\mathsf{t} \vdash \gamma_1 + \gamma_2} \supset T \qquad \frac{\Gamma, A\,\mathsf{t}, B\,\mathsf{f} \vdash \gamma}{\Gamma, A \supset B\,\mathsf{f} \vdash \gamma} \supset F$$

Where $+$ is defined by

$$\cdot + \cdot = \cdot$$
$$C + \cdot = \cdot$$
$$\cdot + C = \cdot$$

where $C_1 + C_2$ is undefined even if $C_1 = C_2$. That is, $\supset T$ is really three rules where one or both (but not neither) of the premises have an empty right-hand side.

We may ask, does cut elimination hold for this calculus? Is the rule

$$\frac{\Gamma \vdash A \qquad \Gamma, A \vdash \gamma}{\Gamma \vdash \gamma} \, cut$$

admissible? In the event that A is of the form $\mathbf{C}B$ and the derivations of $\Gamma \vdash \mathbf{C}B$ and $\Gamma, \mathbf{C}B \vdash \gamma$ both end with a rule that introduces the connective in the cut formula, the derivation looks like

$$\frac{\dfrac{\Gamma, B\,\mathsf{f} \vdash \cdot}{\Gamma \vdash \mathbf{C}B} \mathbf{C}R \qquad \dfrac{\Gamma, B\,\mathsf{t} \vdash \gamma}{\Gamma, \mathbf{C}B \vdash \gamma} \mathbf{C}L}{\Gamma \vdash \gamma} \, cut$$

In order to proceed, it seems necessary that we also show the admissibility of a classical cut principle

$$\frac{\Gamma, B\,\mathsf{f} \vdash \cdot \qquad \Gamma, B\,\mathsf{t} \vdash \gamma}{\Gamma \vdash \gamma} \, sync$$

where false assumptions $A\,\mathsf{f}$ (thought of as continuations, or 'A readers') synchronize with true assumptions $A\,\mathsf{t}$ (thought of as 'A writers'). But proving this rule admissible is problematic. Consider the left-derivation true-commutative case for $\supset$:

$$\frac{\dfrac{\Gamma, A_1\,\mathsf{f}, B\,\mathsf{f} \vdash \cdot \qquad \Gamma, A_2\,\mathsf{t}, B\,\mathsf{f} \vdash \cdot}{\Gamma, A_1, \supset A_2\,\mathsf{t}, B\,\mathsf{f} \vdash \cdot} \qquad \Gamma, A_1 \supset A_2\,\mathsf{t}, B\,\mathsf{t} \vdash \gamma}{\Gamma, A_1 \supset A_2\,\mathsf{t} \vdash \gamma} \, sync$$

We can sync $\Gamma, A_2\,\mathsf{t}, B\,\mathsf{f} \vdash \cdot$ with $\Gamma, A_1 \supset A_2\,\mathsf{t}, B\,\mathsf{t} \vdash \gamma$ (on $B\,\mathsf{t}, B\,\mathsf{f}$) to obtain $\Gamma, A_2\,\mathsf{t}, A_1 \supset A_2\,\mathsf{t} \vdash \gamma$. We can sync $\Gamma, A_1\,\mathsf{f}, B\,\mathsf{f} \vdash \cdot$ with $\Gamma, A_1 \supset A_2\,\mathsf{t}, B\,\mathsf{t} \vdash \gamma$ (on $B\,\mathsf{t}, B\,\mathsf{f}$) to obtain $\Gamma, A_1\,\mathsf{f}, A_1 \supset A_2\,\mathsf{t} \vdash \gamma$.

We would like to derive

$$\frac{\Gamma, A_1\,\mathsf{f}, A_1 \supset A_2\,\mathsf{t} \vdash \gamma \qquad \Gamma, A_2\,\mathsf{t}\, A_1 \supset A_2\,\mathsf{t} \vdash \gamma}{\Gamma, A_1 \supset A_2\,\mathsf{t}, A_1 \supset A_2\,\mathsf{t} \vdash \gamma} \supset T$$

and apply contraction, but it may not be that $\gamma + \gamma = \gamma$! I still haven't found a concrete counterexample, however. So the question is still open:

Question Are the rules cut and sync admissible?