The Identity Theorem in Hereditary Spine Form

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January 24, 2007

Terms	M	$::=$	$x \cdot S \mid \lambda x.S$
Spines	S	$::=$	$() \mid (M; S)$
Syntactic Objects	X	$::=$	$M \mid S$

Eta-expansion:

$$\begin{aligned} \mathsf{ex}_a(H \cdot S) &= H \cdot S \\ \mathsf{ex}_{A \to B}(H \cdot S) &= \lambda y.\mathsf{ex}_B(H \cdot (S; \mathsf{ex}_A(x \cdot ()))) \end{aligned}$$

Substitution and reduction are as usual, abbreviating $\sigma = [M/x]^A$:

$$\begin{aligned} \sigma(\lambda y.N) &= \lambda y.\sigma N \\ \sigma(y \cdot S) &= y \cdot \sigma S \\ \sigma(x \cdot S) &= [M \mid \sigma S] \\ \sigma() &= () \\ \sigma(N; S) &= (\sigma N; \sigma S) \\ [\lambda x.N \mid (M; S)]^{A \to B} &= [[M/x]^A N \mid S]^B \\ [x \cdot S \mid ()]^a &= x \cdot S \\ [M \| ()]^A &= M \end{aligned}$$

In all the following lemmas, $M \Rightarrow N$ means that if M is defined, then N is defined, and $M = N$.

0.1 Lemmas

Lemma 0.1 $[M \| (S; S')]^B \Rightarrow [[M \| S]^{B_2} \| S']^{B_2}$ *for some* B_1, B_2.

Proof Induction on S.

Case: $S = ()$. Immediate by definition of reduction. Pick $B_1 = a$ and $B_2 = B$.

Case: $S = (M_0; S_0)$. Then M is of the form $\lambda y.N$, and B must be of the form $B_L \to B_R$.

$$\begin{aligned} [M \| (S; S')]^B &= [\lambda y.N \| (M_0; S_0; S')]^B \\ &= [[M_0/y]^{B_L} N \| (S_0; S')]^{B_R} \\ &= [[[M_0/y]^{B_L} N \| S_0]^{B_{R1}} \| S']^{B_{R2}} && \text{by i.h.} \\ &= [[\lambda y.N \| (M_0; S_0)]^{B_L \to B_{R1}} \| S']^{B_{R2}} \\ &= [[M \| S]^{B_L \to B_{R1}} \| S']^{B_{R2}} \end{aligned}$$

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Lemma 0.2 $[M/x]^A \mathsf{ex}_B(y \cdot S) \Rightarrow \mathsf{ex}_B(y \cdot [M/x]^A S)$.

Proof By induction on B.

Case: $B = a$. Immediate by definition of ex and substitution.

Case: $B = B_1 \to B_2$. Then

$$\begin{array}{ll}
[M/x]^A \mathsf{ex}_{B_1 \to B_2}(y \cdot S) & \\
= [M/x]^A \lambda z.\mathsf{ex}_{B_2}(y \cdot (S; \mathsf{ex}_{B_1}(z \cdot ()))) & \\
= \lambda z.[M/x]^A \mathsf{ex}_{B_2}(y \cdot (S; \mathsf{ex}_{B_1}(z \cdot ()))) & \\
= \lambda z.\mathsf{ex}_{B_2}(y \cdot [M/x]^A (S; \mathsf{ex}_{B_1}(z \cdot ()))) & \text{by i.h. on } B_2 \\
= \lambda z.\mathsf{ex}_{B_2}(y \cdot ([M/x]^A S; [M/x]^A \mathsf{ex}_{B_1}(z \cdot ()))) & \\
= \lambda z.\mathsf{ex}_{B_2}(y \cdot ([M/x]^A S; \mathsf{ex}_{B_1}([M/x]^A (z \cdot ())))) & \text{by i.h. on } B_1 \\
= \lambda z.\mathsf{ex}_{B_2}(y \cdot ([M/x]^A S; \mathsf{ex}_{B_1}(z \cdot ()))) & \\
= \mathsf{ex}_{B_1 \to B_2}(y \cdot ([M/x]^A S) &
\end{array}$$

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Lemma 0.3

1. $[\mathsf{ex}_A(x \cdot ())/y]^B X \Rightarrow [x/y]X$
2. $[\mathsf{ex}_A(x \cdot S) \mid S']^B \Rightarrow x \cdot (S; S')$
3. $[M/x]^B \mathsf{ex}_A(x \cdot S) \Rightarrow [M \| S]^{B'}$, *if* $x \notin FV(S)$, *for some* B'.

Proof By lexicographic induction on A, the case 1–3, and the object X.

1. Split cases on the structure of X. The reasoning is straightforward except when $X = y \cdot S$. Then we must show, by the definition of substitution,

$$[\mathsf{ex}_A(x \cdot ()) \mid [\mathsf{ex}_A(x \cdot ())/y]^B S]^B \Rightarrow x \cdot ([x/y]S)$$

   We get $[\mathsf{ex}_A(x \cdot ())/y]^B S = [x/y]S$ from the i.h. part 1, on the smaller expression S. Then appeal to the i.h. part 2 on the same type A to see that

$$[\mathsf{ex}_A(x \cdot ()) \mid [x/y]S]^B \Rightarrow x \cdot ([x/y]S)$$

2. Split cases on A.

   Case: $A = a$. Immediate from definitions. Note that S' must be $()$ and B must be a base type for the left hand side to be defined.

   Case: $A = A_1 \to A_2$. S' must be of the form $(M_0; S_0)$, and B must be of the form $B_1 \to B_2$.

$$
\begin{aligned}
&[\mathsf{ex}_{A_1\to A_2}(x\cdot S) \mid S']^{B_1\to B_2} \\
&= [\lambda y.\mathsf{ex}_{A_2}(x\cdot(S;\mathsf{ex}_{A_1}(y\cdot()))) \mid (M_0;S_0)]^{B_1\to B_2} \\
&= [[M_0/y]^{B_1}\mathsf{ex}_{A_2}(x\cdot(S;\mathsf{ex}_{A_1}(y\cdot()))) \mid S_0]^{B_2} \\
&= [\mathsf{ex}_{A_2}(x\cdot[M_0/y]^{B_1}(S;\mathsf{ex}_{A_1}(y\cdot()))) \mid S_0]^{B_2} && \text{by Lemma 0.2} \\
&= [\mathsf{ex}_{A_2}(x\cdot(S;[M_0/y]^{B_1}\mathsf{ex}_{A_1}(y\cdot()))) \mid S_0]^{B_2} && y\notin FV(S) \\
&= [\mathsf{ex}_{A_2}(x\cdot(S;[M_0\|()]^{B'})) \mid S_0]^{B_2} && \text{by i.h. 3 on } A_1 \\
&= [\mathsf{ex}_{A_2}(x\cdot(S;M_0)) \mid S_0]^{B_2} \\
&= x\cdot(S;M_0;S_0) && \text{by i.h. 2 on } A_2 \\
&= x\cdot(S;S')
\end{aligned}
$$

3. Split cases on A.

Case: $A = a$. Immediate from assumption and definitions.

Case: $A = A_1 \to A_2$. Note first of all that for any N, C that

$$\lambda y.[N\|(\mathsf{ex}_{A_1}(y\cdot()))]^C \Rightarrow N$$

This is because N must be of the form $\lambda z.N_0$ and C of the form $C_1 \to C_2$ for the left-hand side to be defined, in which case we have

$$
\begin{aligned}
&\lambda y.[\lambda z.N_0\|(\mathsf{ex}_{A_1}(y\cdot()))]^{C_1\to C_2} \\
&= \lambda y.[[\mathsf{ex}_{A_1}(y\cdot())/z]^{C_1}N_0\|()]^{C_2} \\
&= \lambda y.[\mathsf{ex}_{A_1}(y\cdot())/z]^{C_1}N_0 \\
&= \lambda y.[y/z]N_0 && \text{by i.h. 1 on } A_1 \\
&= N && \alpha\text{-equivalence}
\end{aligned}
$$

Having made this observation, compute

$$
\begin{aligned}
&[M/x]^B\mathsf{ex}_{A_1\to A_2}(x\cdot S) \\
&= [M/x]^B\lambda y.\mathsf{ex}_{A_2}(x\cdot(S;\mathsf{ex}_{A_1}(y\cdot()))) \\
&= \lambda y.[M/x]^B\mathsf{ex}_{A_2}(x\cdot(S;\mathsf{ex}_{A_1}(y\cdot()))) \\
&= \lambda y.[M\|(S;\mathsf{ex}_{A_1}(y\cdot()))]^{B'} && \text{by i.h. 3 on } A_2 \\
&= \lambda y.[[M\|S]^{B_1}\|(\mathsf{ex}_{A_1}(y\cdot()))]^{B_2} && \text{by Lemma 0.1} \\
&= [M\|S]^{B_1} && \text{by above observation}
\end{aligned}
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