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Choose a universe V. and a universe U such that $\mathsf{V} \in \mathsf{U}$. Let $\mathbf{Fcc}$ be the category of finitely complete U-small categories with functors preserving finite limits as arrows. An *internal category* in a finitely complete category $\mathbf{C}$ is an object H of $\mathbf{C}$ and arrows $d_0, d_1 \colon H \to H$ and $\mathsf{comp} \colon H \times_H H \to H$ — where $H \times_H H$ is the pullback

$$\begin{tikzcd}
H \times_H H \arrow[r, "\mathrm{snd}"] \arrow[d, "\mathrm{fst}"'] \arrow[dr, phantom, "\lrcorner", very near start] & H \arrow[d, "d_0"] \\
H \arrow[r, "d_1"'] & H
\end{tikzcd}$$

of d_1 and d_0 — such that the following diagrams commute:

$$(C1) \qquad \begin{tikzcd}
H \arrow[dr, "d_1"] & \\
H \arrow[u, "d_0"] \arrow[r, "d_0"'] & H
\end{tikzcd}
\qquad
\begin{tikzcd}
H \arrow[dr, "d_0"] & \\
H \arrow[u, "d_1"] \arrow[r, "d_1"'] & H
\end{tikzcd} \qquad (C2)$$

$$(C3) \qquad \begin{tikzcd}
H \arrow[r, "\langle \mathrm{id}_H, d_1 \rangle_H"] \arrow[dr, equal] & H \times_H H \arrow[d, "\mathsf{comp}"] & H \arrow[l, "\langle d_0, \mathrm{id}_H \rangle_H"'] \arrow[dl, equal] \\
& H &
\end{tikzcd}$$

$$(C4) \qquad \begin{tikzcd}
H \arrow[d, "d_0"'] & H \times_H H \arrow[l, "\mathrm{fst}"'] \arrow[r, "\mathrm{snd}"] \arrow[d, "\mathsf{comp}"] & H \arrow[d, "d_1"] \\
H & H \arrow[l, "d_0"] \arrow[r, "d_1"'] & H
\end{tikzcd}$$

$$(C5) \qquad \begin{tikzcd}
H \times_H H \times_H H \arrow[r, "\mathrm{id}_H \times_H \mathsf{comp}"] \arrow[d, "\mathsf{comp} \times_H \mathrm{id}_H"'] & H \times_H H \arrow[d, "\mathsf{comp}"] \\
H \times_H H \arrow[r, "\mathsf{comp}"'] & H
\end{tikzcd}$$

An *internal functor* from an internal category $(H, d_0, d_1, \mathsf{comp})$ in $\mathbf{C}$ to an internal category $(H', d_0', d_1', \mathsf{comp}')$

in $\mathbf{C}$ is an arrow $F\colon H \to H'$ in $\mathbf{C}$ such that the following diagrams commute:

$$(F1) \qquad \begin{tikzcd} H \arrow[r, "F"] & H' \\ H \arrow[u, "d_0"] \arrow[r, "F"] \arrow[d, "d_1"'] & H' \arrow[u, "d'_0"'] \arrow[d, "d'_1"] \\ H \arrow[r, "F"] & H' \end{tikzcd}$$

$$(F2) \qquad \begin{tikzcd} H \times_H H \arrow[r, "\langle F \circ \mathrm{fst}, F \circ \mathrm{snd} \rangle_{H'}"] \arrow[d, "\mathsf{comp}"'] & H' \times_{H'} H' \arrow[d, "\mathsf{comp}'"] \\ H \arrow[r, "F"'] & H' \end{tikzcd}$$

▷ Proposition: Internal categories and internal functors in a finitely complete U-small category $\mathbf{C}$ form a U-small category, $\mathsf{Int}(\mathbf{C})$.

Proof: It clearly follows from the fact that $\mathbf{C}$ is U-small that the collections of internal categories in $\mathbf{C}$ and internal functors are respectively U-small. We define composition of internal functors in $\mathbf{C}$ by composition of the corresponding arrows in $\mathbf{C}$.

• Composition is well-defined:

If we have internal functors $F\colon H \to H'$ and $G\colon H' \to H''$ in $\mathbf{C}$, then certainly $G \circ F$ is an arrow of $\mathbf{C}$, but we must check that it is an internal functor $H \to H''$. Preservation of 'domain' and 'codomain' is an easy diagram-chase:

$$\begin{tikzcd} H \arrow[r, "F"] & H' \arrow[r, "G"] & H'' \\ H \arrow[u, "d_0"] \arrow[r, "F"] \arrow[d, "d_1"'] & H' \arrow[u, "d'_0"'] \arrow[r, "G"] \arrow[d, "d'_1"] & H'' \arrow[u, "d''_0"'] \arrow[d, "d''_1"] \\ H \arrow[r, "F"] & H' \arrow[r, "G"] & H'' \end{tikzcd}$$

To see that 'composition' is preserved for the composite we must establish the commutativity of

$$\begin{tikzcd} H \times_H H \arrow[r, "\langle GF\mathrm{fst}, GF\mathrm{snd} \rangle_{H''}"] \arrow[d, "\mathsf{comp}"'] & H'' \times_{H''} H'' \arrow[d, "\mathsf{comp}''"] \\ H \arrow[r, "GF"'] & H'' \end{tikzcd}$$

given the commutativity of

$$\begin{tikzcd}
H \times_H H \arrow[r, "\langle F\mathrm{fst}, F\mathrm{snd}\rangle_{H'}"] \arrow[d, "\mathsf{comp}"'] & H' \times_{H'} H' \arrow[r, "\langle G\mathrm{fst}', G\mathrm{snd}'\rangle_{H''}"] \arrow[d, "\mathsf{comp}'"] & H'' \times_{H''} H'' \arrow[d, "\mathsf{comp}''"] \\
H \arrow[r, "F"'] & H' \arrow[r, "G"'] & H''
\end{tikzcd}$$

so it suffices to show that $\langle G\mathrm{fst}', G\mathrm{snd}'\rangle_{H''}\langle F\mathrm{fst}, F\mathrm{snd}\rangle_{H'} = \langle GF\mathrm{fst}, GF\mathrm{snd}\rangle_{H''}$. But observe that we have

$$\mathrm{fst}''\langle G\mathrm{fst}', G\mathrm{snd}'\rangle_{H''}\langle F\mathrm{fst}, F\mathrm{snd}\rangle_{H'} = G\mathrm{fst}'\langle F\mathrm{fst}, F\mathrm{snd}\rangle_{H'} = GF\mathrm{fst}$$

and

$$\mathrm{snd}''\langle G\mathrm{fst}', G\mathrm{snd}'\rangle_{H''}\langle F\mathrm{fst}, F\mathrm{snd}\rangle_{H'} = G\mathrm{snd}'\langle F\mathrm{fst}, F\mathrm{snd}\rangle_{H'} = GF\mathrm{snd}$$

so that $\langle G\mathrm{fst}', G\mathrm{snd}'\rangle_{H''}\langle F\mathrm{fst}, F\mathrm{snd}\rangle_{H'}$ has the property which uniquely characterizes $\langle GF\mathrm{fst}, GF\mathrm{snd}\rangle_{H''}$, namely that it fits into the pullback diagram

$$\begin{tikzcd}
H \times_H H \arrow[dr, dotted] \arrow[drr, bend left, "GF\mathrm{snd}"] \arrow[ddr, bend right, "GF\mathrm{fst}"'] & & \\
& H'' \times_{H''} H'' \arrow[r, "\mathrm{snd}''"'] \arrow[d, "\mathrm{fst}''"] & H'' \arrow[d, "d_0''"] \\
& H'' \arrow[r, "d_1''"'] & H''
\end{tikzcd}$$

• Identities exist:
The identity internal functor for an internal category $(H, d_0, d_1, \mathsf{comp})$ is just the identity arrow on H. We need to check that id_H is an internal functor; preservation of 'domain' and 'codomain' is trivial, and preservation of 'composition' again follows from the universal property of the pullback. That is, we have that $\langle \mathrm{id}_H\mathrm{fst}, \mathrm{id}_H\mathrm{snd}\rangle_H = \mathrm{id}_{H\times_H H}$, so that (F2) commutes for $H' = H$ and $F = \mathrm{id}_H$. That these identites behave correctly under composition follows directly from the behavior of identity arrows in $\mathbf{C}$.
• Composition is associative:
Associativity of composition in $\mathsf{Int}(\mathbf{C})$ is again a direct conseqence of associativity of composition in $\mathbf{C}$. ■

▷ PROPOSITION: Let $\mathbf{C}$ be a finitely complete U-small category. Then $\mathsf{Int}(\mathbf{C})$ is finitely complete.
PROOF:
• Terminal object:
The terminal object of $\mathsf{Int}(\mathbf{C})$ is the internal category whose 'object of homomorphisms' H is the terminal object $1_{\mathbf{C}}$ of $\mathbf{C}$. The arrows d_0, d_1, comp are all then necessarily the unique endomorphism $!\colon 1_{\mathbf{C}} \to 1_{\mathbf{C}}$. The universal property of the terminal object of $\mathsf{Int}(\mathbf{C})$ is satisfied precisely because it is satisfied for the terminal object of $\mathbf{C}$.
• Pairwise products:
Suppose $(H, d_0, d_1, \mathsf{comp}), (H', d_0', d_1', \mathsf{comp}')$ are two internal categories in $\mathbf{C}$. We define the product in $\mathsf{Int}(\mathbf{C})$ 'pointwise', i.e. $(H \times H', d_0 \times d_0', d_1 \times d_1', \mathsf{comp} \times \mathsf{comp}')$. There appears to be a type mismatch; the type of $\mathsf{comp}\times\mathsf{comp}'$ is $(H\times_H H)\times(H'\times_{H'} H') \to H\times H'$, and we want it to be $(H\times H')\times_{H\times H'}(H\times H') \to H\times H'$. However, $(H \times H') \times_{H\times H'} (H \times H')$ and $(H \times_H H) \times (H' \times_{H'} H')$ are uniquely isomorphic since $\times$ is a right adjoint and so preserves limits, in particular pullback diagrams. That $H \times H'$ *et al.* is in fact an internal category follows from further application of preservation of limits by the product. It remains to check that

the universal property of the product holds; suppose we have an internal category $(H'', d_0'', d_1'', \mathsf{comp}'')$ and internal functors $F\colon H'' \to H$ and $G\colon H'' \to H'$. Now $\langle F, G\rangle$ (where pairing is in $\mathbf{C}$) is the unique morphism in $\mathbf{C}$ such that $\mathsf{L}\langle F, G\rangle = F$ and $\mathsf{R}\langle F, G\rangle = G$, where L, R are the projections from $H \times H'$, again in $\mathbf{C}$. It is easy to check that these projections are in fact internal functors, and that the pairing is an internal functor.
• Equalizers:
Suppose we have two internal parallel functors

$$\begin{tikzcd} H \arrow[r, shift left, "F"] \arrow[r, shift right, "G"'] & H' \end{tikzcd}$$

we can find the equalizer in $\mathbf{C}$ to obtain an object H'' and an arrow $E\colon H'' \to H$ in $\mathbf{C}$:

$$\begin{tikzcd} H'' \arrow[r, "E"] & H \arrow[r, shift left, "F"] \arrow[r, shift right, "G"'] & H' \end{tikzcd}$$

Our first priority is to find structure to make H'' an internal category. Consider the diagram

$$\begin{tikzcd} H'' \arrow[r, "E"] & H \arrow[r, shift left, "F"] \arrow[r, shift right, "G"'] & H' \\ H'' \arrow[r, "E"'] & H \arrow[u, "d_0"] \arrow[r, shift left, "F"] \arrow[r, shift right, "G"'] & H' \arrow[u, "d_0'"'] \end{tikzcd}$$

and notice that since F, G are internal functors and E equalizes F, G, we have $Fd_0E = d_0'FE = d_0'GE = Gd_0E$. That is, d_0E also equalizes F, G, which provides us an arrow $d_0''\colon H'' \to H''$ such that the following diagram commutes:

$$\begin{tikzcd} H'' \arrow[r, "E"] & H \arrow[r, shift left, "F"] \arrow[r, shift right, "G"'] & H' \\ H'' \arrow[u, dotted, "d_0''"] \arrow[r, "E"'] & H \arrow[u, "d_0"] \arrow[r, shift left, "F"] \arrow[r, shift right, "G"'] & H' \arrow[u, "d_0'"'] \end{tikzcd}$$

Similarly, we get d_1'' out of the diagram

$$\begin{tikzcd} H'' \arrow[r, "E"] & H \arrow[r, shift left, "F"] \arrow[r, shift right, "G"'] & H' \\ H'' \arrow[u, dotted, "d_1''"] \arrow[r, "E"'] & H \arrow[u, "d_1"] \arrow[r, shift left, "F"] \arrow[r, shift right, "G"'] & H' \arrow[u, "d_1'"'] \end{tikzcd}$$

since d_1E equalizes F, G, and we get comp'' out of the diagram

$$\begin{tikzcd} H'' \arrow[r, "E"] & H \arrow[r, shift left, "F"] \arrow[r, shift right, "G"'] & H' \\ H'' \times_{H''} H'' \arrow[u, dotted, "\mathsf{comp}''"] \arrow[r, "\langle E\mathrm{fst}'', E\mathrm{snd}''\rangle_H"'] & H \times_H H \arrow[u, "\mathsf{comp}"] \arrow[r, shift left, "\langle F\mathrm{fst}, F\mathrm{snd}\rangle_{H'}"] \arrow[r, shift right, "\langle G\mathrm{fst}, G\mathrm{snd}\rangle_{H'}"'] & H' \times_{H'} H' \arrow[u, "\mathsf{comp}'"'] \end{tikzcd}$$

and observing that

$$\begin{aligned}
\langle F\mathrm{fst}, F\mathrm{snd}\rangle_{H'}\langle E\mathrm{fst}'', E\mathrm{snd}''\rangle_H &= \langle FE\mathrm{fst}'', FE\mathrm{snd}''\rangle_H \\
&= \langle GE\mathrm{fst}'', GE\mathrm{snd}''\rangle_H \\
&= \langle G\mathrm{fst}, G\mathrm{snd}\rangle_{H'}\langle E\mathrm{fst}'', E\mathrm{snd}''\rangle_H
\end{aligned}$$

Now it is easy to check that $(H'', d_0'', d_1'', \mathsf{comp}'')$ is an internal category; for example, by choice d_0'' is the unique arrow such that $Ed_0'' = d_0E$, yet $Ed_1''d_0'' = d_1Ed_0'' = d_1d_0E = d_0E$, so it must be that $d_1''d_0'' = d_0''$. It falls out of the diagrams already shown that E is an internal functor $H'' \to H$. Now suppose we have a situation as depicted in the diagram

$$\begin{tikzcd}
H'' \arrow[r, "E"] & H \arrow[r, shift left, "F"] \arrow[r, shift right, "G"'] & H' \\
 & H^{(3)} \arrow[u, "I"'] \arrow[ul, dotted, "J"] &
\end{tikzcd}$$

where $H^{(3)}$ is an internal category and I is an internal functor equalizing F, G. The universal property of the equalizer in $\mathbf{C}$ guarantees a unique arrow J making the diagram commute. To see that J is internal functorial, consider the following diagram:

$$\begin{tikzcd}
H'' \arrow[rrr, "d_0''"] \arrow[dr, "E"] & & & H'' \arrow[dl, "E"'] \\
 & H \arrow[r, "d_0"] & H & \\
H^{(3)} \arrow[uu, "J"] \arrow[ur, "I"] \arrow[rrr, "d_0^{(3)}"'] & & & H^{(3)} \arrow[uu, "J"'] \arrow[ul, "I"']
\end{tikzcd}$$

the internal trapezoids commute because E, I are internal functors, and the triangles commute by choice of J. Hence $Ed_0''J = EJd_0^{(3)}$, and because E (being an equalizer) is mono, we know $d_0''J = Jd_0^{(3)}$. The proofs that J preserves 'codomain' and 'composition' are similar. Therefore we conclude that E is the equalizer of F, G in $\mathsf{Int}(\mathbf{C})$. ∎

▷ PROPOSITION: $\mathbf{C} \mapsto \mathsf{Int}(\mathbf{C})$ as defined above is the object part of a functor $\mathsf{Int}\colon \mathbf{Fcc} \to \mathbf{Fcc}$.

PROOF: Let $\mathbf{F}\colon \mathbf{C} \to \mathbf{D}$ be an arrow in $\mathbf{Fcc}$, i.e. a finite limit preserving functor from $\mathbf{C}$ to $\mathbf{D}$. Now $\mathsf{Int}(\mathbf{F})$ also needs to be a finite limit preserving functor, but from $\mathsf{Int}(\mathbf{C})$ to $\mathsf{Int}(\mathbf{D})$. Suppose we have an internal category $(H, d_0, d_1, \mathsf{comp})$ in $\mathbf{C}$. Then we define $\mathsf{Int}(\mathbf{F})(H, d_0, d_1, \mathsf{comp}) := (\mathbf{F}H, \mathbf{F}d_0, \mathbf{F}d_1, \mathbf{F}\mathsf{comp})$. Because $\mathbf{F}$ preserves composition and finite products, $(\mathbf{F}H, \mathbf{F}d_0, \mathbf{F}d_1, \mathbf{F}\mathsf{comp})$ is an internal category in $\mathbf{D}$. The action of $\mathsf{Int}(\mathbf{F})$ on arrows of $\mathsf{Int}(\mathbf{C})$ is the obvious one; $\mathsf{Int}(\mathbf{F})(F) := \mathbf{F}F$. Now it is possible to see that $\mathsf{Int}(\mathbf{F})$ is functorial since $\mathsf{Int}(\mathbf{F})(\mathrm{id}_H) = \mathbf{F}(\mathrm{id}_H) = \mathrm{id}_{\mathbf{F}H} = \mathrm{id}_{\mathsf{Int}(\mathbf{F})(H)}$ for any internal category object H in $\mathbf{C}$, and $\mathsf{Int}(\mathbf{F})(G \circ F) = \mathbf{F}(G \circ F) = \mathbf{F}(G) \circ \mathbf{F}(F) = \mathsf{Int}(\mathbf{F})(G) \circ \mathsf{Int}(\mathbf{F})(F)$ for any internal functors $F\colon H \to H'$ and $G\colon H' \to H''$ in $\mathbf{C}$. Now we check that $\mathsf{Int}(\mathbf{F})$ preserves finite limits; We have first of all that $\mathsf{Int}(\mathbf{F})(1_{\mathsf{Int}(\mathbf{C})}) = (\mathbf{F}1_{\mathsf{Int}(\mathbf{C})}, \mathbf{F}!, \mathbf{F}!, \mathbf{F}!) = (1_{\mathsf{Int}(\mathbf{D})}, !, !, !)$ because $\mathbf{F}$ preserves the terminal object. Moreover, just by unrolling definitions again we see

$$\begin{aligned}
\mathsf{Int}(\mathbf{F})((H, d_0, d_1, \mathsf{comp}) \times (H', d_0', d_1', \mathsf{comp}')) &= \mathsf{Int}(\mathbf{F})(H \times H', d_0 \times d_0', d_1 \times d_1', \mathsf{comp} \times \mathsf{comp}') \\
&= (\mathbf{F}(H \times H'), \mathbf{F}(d_0 \times d_0'), \mathbf{F}(d_1 \times d_1'), \mathbf{F}(\mathsf{comp} \times \mathsf{comp}')) \\
&= (\mathbf{F}H \times \mathbf{F}H', \mathbf{F}d_0 \times \mathbf{F}d_0', \mathbf{F}d_1 \times \mathbf{F}d_1', \mathbf{F}\mathsf{comp} \times \mathbf{F}\mathsf{comp}') \\
&= (\mathbf{F}H, \mathbf{F}d_0, \mathbf{F}d_1, \mathbf{F}\mathsf{comp}) \times (\mathbf{F}H', \mathbf{F}d_0', \mathbf{F}d_1', \mathbf{F}\mathsf{comp}') \\
&= (\mathbf{F}H, \mathbf{F}d_0, \mathbf{F}d_1, \mathbf{F}\mathsf{comp}) \times (\mathbf{F}H', \mathbf{F}d_0', \mathbf{F}d_1', \mathbf{F}\mathsf{comp}') \\
&= \mathsf{Int}(\mathbf{F})(H, d_0, d_1, \mathsf{comp}) \times \mathsf{Int}(\mathbf{F})(H', d_0', d_1', \mathsf{comp}')
\end{aligned}$$

and similarly $\mathsf{Int}(\mathbf{F})$ preserves equalizers because $\mathbf{F}$ does. Finally, we want Int itself to be functorial. Suppose we have arrows $\mathbf{F}\colon \mathbf{C} \to \mathbf{D}$ and $\mathbf{G}\colon \mathbf{D} \to \mathbf{E}$ in $\mathbf{Fcc}$. Then

$$\begin{aligned}
\mathsf{Int}(\mathbf{G} \circ \mathbf{F})(H, d_0, d_1, \mathsf{comp}) &= ((\mathbf{G} \circ \mathbf{F})(H), (\mathbf{G} \circ \mathbf{F})(d_0), (\mathbf{G} \circ \mathbf{F})(d_1), (\mathbf{G} \circ \mathbf{F})(\mathsf{comp})) \\
&= (\mathbf{G}(\mathbf{F}(H)), \mathbf{G}(\mathbf{F}(d_0)), \mathbf{G}(\mathbf{F}(d_1)), \mathbf{G}(\mathbf{F}(\mathsf{comp}))) \\
&= \mathsf{Int}(\mathbf{G})(\mathbf{F}(H), \mathbf{F}(d_0), \mathbf{F}(d_1), \mathbf{F}(\mathsf{comp})) \\
&= \mathsf{Int}(\mathbf{G})(\mathsf{Int}(\mathbf{F})(H, d_0, d_1, \mathsf{comp})) \\
&= (\mathsf{Int}(\mathbf{G}) \circ \mathsf{Int}(\mathbf{F}))(H, d_0, d_1, \mathsf{comp})
\end{aligned}$$

and for any object $\mathbf{C}$ of $\mathbf{Fcc}$ we know

$$\begin{aligned}
\mathsf{Int}(\mathrm{id}_{\mathbf{C}})(H, d_0, d_1, \mathsf{comp}) &= (\mathrm{id}_{\mathbf{C}}(H), \mathrm{id}_{\mathbf{C}}(d_0), \mathrm{id}_{\mathbf{C}}(d_1), \mathrm{id}_{\mathbf{C}}(\mathsf{comp})) \\
&= (H, d_0, d_1, \mathsf{comp})
\end{aligned}$$

and so $\mathsf{Int}(\mathrm{id}_{\mathbf{C}}) = \mathrm{id}_{\mathsf{Int}(\mathbf{C})}$, hence $\mathsf{Int}\colon \mathbf{Fcc} \to \mathbf{Fcc}$ is functorial. ∎

▷ Conjecture: There exists a functor $\mathsf{Ext}\colon \mathbf{Fcc} \to \mathbf{Fcc}$ such that $\mathsf{Int} \dashv \mathsf{Ext}$.

▷ Conjecture: $\mathbf{Fcc}$ is countably cocomplete.

▷ Corollary: There exists a finitely complete U-small category $\mathbf{C}$ such that $\mathsf{Int}(\mathbf{C}) \cong \mathbf{C}$.

Proof: Let $\mathbf{Sets}_{\mathsf{V}}$ be the (U-small) category of all V-small sets. There exists a functor $F\colon \mathbf{Sets}_{\mathsf{V}} \to \mathsf{Int}(\mathbf{Sets}_{\mathsf{V}})$ defined by

$$F(S) := (S, \mathrm{id}_S, \mathrm{id}_S, \mathrm{id}_S)$$

and

$$F(f) := f$$

So we can form the diagram

$$D \quad := \quad \mathbf{Sets}_{\mathsf{V}} \xrightarrow{f} \mathsf{Int}(\mathbf{Sets}_{\mathsf{V}}) \xrightarrow{\mathsf{Int}(f)} \mathsf{Int}^2(\mathbf{Sets}_{\mathsf{V}}) \xrightarrow{\mathsf{Int}^2(f)} \cdots$$

and set $\mathbf{C} := \varinjlim D$ to obtain

$$\begin{tikzcd}
\mathbf{Sets}_{\mathsf{V}} \arrow[r, "f"] \arrow[dr, "i_0"'] & \mathsf{Int}(\mathbf{Sets}_{\mathsf{V}}) \arrow[r, "\mathsf{Int}(f)"] \arrow[d, "i_1"] & \mathsf{Int}^2(\mathbf{Sets}_{\mathsf{V}}) \arrow[r, "\mathsf{Int}^2(f)"] \arrow[dl, "i_2"] & \cdots \\
& \mathbf{C} & \cdots &
\end{tikzcd}$$

Now Int, being a left adjoint, preserves colimits, so we have that

$$\begin{tikzcd}
\mathsf{Int}(\mathbf{Sets}_{\mathsf{V}}) \arrow[r, "\mathsf{Int}(f)"] \arrow[dr, "\mathsf{Int}(i_0)"'] & \mathsf{Int}^2(\mathbf{Sets}_{\mathsf{V}}) \arrow[r, "\mathsf{Int}^2(f)"] \arrow[d, "\mathsf{Int}(i_1)"] & \mathsf{Int}^3(\mathbf{Sets}_{\mathsf{V}}) \arrow[r, "\mathsf{Int}^3(f)"] \arrow[dl, "\mathsf{Int}(i_2)"] & \cdots \\
& \mathsf{Int}(\mathbf{C}) & \cdots &
\end{tikzcd}$$

is also a colimit diagram. That is, $\mathsf{Int}(\mathbf{C}) = \varinjlim \mathsf{Int}(D)$. Now if we can show that $\mathbf{C}$ is the colimit of $\mathsf{Int}(D)$, then $\mathbf{C} \cong \mathsf{Int}(\mathbf{C})$. Certainly $\mathbf{C}$ together with $i_1, i_2, i_3, \ldots$ is a cocone over $\mathsf{Int}(D)$, so there exists a unique $h\colon \mathsf{Int}(\mathbf{C}) \to \mathbf{C}$ such that

$$\begin{tikzcd}
& \mathsf{Int}^{n+1}(\mathbf{Sets}_{\mathsf{V}}) \arrow[dl, "\mathsf{Int}(i_n)"'] \arrow[dr, "i_{n+1}"] & \\
\mathsf{Int}(\mathbf{C}) \arrow[rr, "h"'] & & \mathbf{C}
\end{tikzcd}$$

commutes for every $n \in \mathbb{N}$. Also $\mathsf{Int}(\mathbf{C})$ together with $\mathsf{Int}(i_0) \circ f, \mathsf{Int}(i_0), \mathsf{Int}(i_1), \mathsf{Int}(i_2), \ldots$ is a cocone over $\mathbf{D}$, so there exists a unique $k\colon \mathbf{C} \to \mathsf{Int}(\mathbf{C})$ such that

$$\begin{tikzcd} & \mathbf{Sets}_{\mathsf{V}} \arrow[dl, "\mathsf{Int}(i_0) \circ f"'] \arrow[dr, "i_0"] & \\ \mathsf{Int}(\mathbf{C}) & & \mathbf{C} \arrow[ll, "k"] \end{tikzcd}$$

commutes and

$$\begin{tikzcd} & \mathsf{Int}^{n+1}(\mathbf{Sets}_{\mathsf{V}}) \arrow[dl, "\mathsf{Int}(i_n)"'] \arrow[dr, "i_{n+1}"] & \\ \mathsf{Int}(\mathbf{C}) & & \mathbf{C} \arrow[ll, "k"] \end{tikzcd}$$

commutes for every $n \in \mathbb{N}$. Notice that h, k are morphisms in the category of cocones over $\mathsf{Int}(D)$, and therefore so too is $k \circ h\colon \mathsf{Int}(\mathbf{C}) \to \mathsf{Int}(\mathbf{C})$. Hence $kh = \mathrm{id}_{\mathsf{Int}(\mathbf{C})}$ since $\mathsf{Int}(\mathbf{C}) = \varinjlim \mathsf{Int}(D)$. But h, k are also morphisms in the category of cocones over D (h is because $h \circ \mathsf{Int}(i_0) \circ f = i_1 \circ f = i_0$) so $hk = \mathrm{id}_{\mathbf{C}}$ since $\mathbf{C} = \varinjlim D$. That is, h, k are an inverse pair, and $\mathbf{C} \cong \mathsf{Int}(\mathbf{C})$. ■