Notes on Split Coequalizers

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Let **Sc** be the category whose objects are $\mathfrak{A}, \mathfrak{B}, 0$ and whose arrows are

$$\delta_0, \delta_1 : \mathfrak{A} \to \mathfrak{B}$$

$$\Delta_0, \Delta_1, \mathrm{id}_{\mathfrak{A}} : \mathfrak{A} \to \mathfrak{A}$$

$$\sigma : \mathfrak{B} \to \mathfrak{A}$$

$$\mathrm{id}_{\mathfrak{B}} : \mathfrak{B} \to \mathfrak{B}$$

$$\{0_{M,N} : M \to N \mid M, N \in \{\mathfrak{A}, \mathfrak{B}, 0\}\}$$

The composition law is given as follows:

$$\delta_0 \sigma = \mathrm{id}_{\mathfrak{B}} \qquad \delta_1 \sigma = 0_{\mathfrak{B}\mathfrak{B}}$$

$$\delta_0 \Delta_i = \delta_i \qquad \delta_1 \Delta_i = 0_{\mathfrak{A}\mathfrak{B}}$$

$$\sigma \delta_i = \Delta_i$$

$$\Delta_0 \Delta_i = \Delta_i \qquad \Delta_1 \Delta_i = 0_{\mathfrak{A}\mathfrak{A}}$$

$$f 0_{M,N} = 0_{M, \mathrm{cod}\, f} \qquad 0_{M,N} f = 0_{\mathrm{dom}\, f, N}$$

Sc sort of looks like this, without all the zeroes:

$$\begin{tikzcd}
\mathfrak{A} \arrow[r, shift left, "\delta_0"] \arrow[r, shift right, "\delta_1"'] & \mathfrak{B} \arrow[d, equal] \\
\mathfrak{A} \arrow[u, shift right, "\Delta_0"] \arrow[u, shift left, "\Delta_1"'] & \mathfrak{B} \arrow[l, "\sigma"]
\end{tikzcd}$$

A *fork* is a diagram

$$\begin{tikzcd}
A \arrow[r, shift left, "d_0"] \arrow[r, shift right, "d_1"'] & B \arrow[r, "e"] & C
\end{tikzcd}$$

such that $ed_0 = ed_1$. A *split fork* is a fork together with a *splitting*

$$\begin{tikzcd} A & B \arrow[l, "s"'] & C \arrow[l, "t"'] \end{tikzcd}$$

such that

$$\begin{tikzcd} B \arrow[r, "s"] \arrow[d, "e"'] & A \arrow[r, "d_0"] \arrow[d, "d_1"'] & B \arrow[d, "e"] \\ C \arrow[r, "t"'] & B \arrow[r, "e"'] & C \end{tikzcd}$$

commutes, and the top two rows compose to identities.

Proposition 0.1 **Sc** *is isomorphic to the category generated by the graph*

$$G \quad := \quad \begin{tikzcd} A \arrow[r, shift left=2, "d_0"] \arrow[r, "d_1" description] & B \arrow[l, shift left=2, "s"] \arrow[r, shift left, "e"] & C \arrow[l, shift left, "t"] \end{tikzcd}$$

with the identities $ed_0 = ed_1$, $d_1 s = te$, $d_0 s = \mathrm{id}_B$, *and* $et = \mathrm{id}_C$.

Proof Let $F\colon \mathbf{Grph} \to \mathbf{Cat}$ be the left adjoint of the forgetful functor $U\colon \mathbf{Cat} \to \mathbf{Grph}$. Let $J = \mathbf{2} + \mathbf{2} + \mathbf{2} + \mathbf{2}$, and define $f_0, f_1 : J \to FG$ by $f_0 := [ed_0, d_1 s, d_0 s, et]$ and $f_1 := [ed_1, te, \mathrm{id}_B, \mathrm{id}_C]$. (That is, each arrow is defined by a copairing of arrows from $\mathbf{2}$ into FG, i.e. particular paths in G) Then Q, the category generated by G subject to the identities is by definition the coequalizer of f_0, f_1.

$$\begin{tikzcd} J \arrow[r, shift left, "f_0"] \arrow[r, shift right, "f_1"'] & FG \arrow[r, "q"] & Q \end{tikzcd}$$

To show $Q \cong \mathbf{Sc}$, it sufficies that $\mathbf{Sc}$ has the universal property of the coequalizer. We need a $r\colon FG \to \mathbf{Sc}$, and for every $h\colon FG \to H$ such that $hf_0 = hf_1$, we want to show that there is a unique k such that

$$\begin{tikzcd} & H \\ FG \arrow[ur, "h"] \arrow[r, "r"'] & \mathbf{Sc} \arrow[u, dashed, "k"'] \end{tikzcd}$$

commutes. Since FG is a free category, we only need to specify r's action on G. The action on objects is obvious: we take A to $\mathfrak{A}$, B to $\mathfrak{B}$, C to 0. The action of arrows is equally apparent: we take d_i to δ_i, s to σ, e to $0_{\mathfrak{B},0}$ and t to $0_{0,\mathfrak{B}}$.

Now if we are to have $kr = h$, then we must have $k(r(x)) = h(x)$ for all $x \in \{d_0, d_1, s, e, t\}$. That is,

$$k(\delta_i) = h(d_i)$$

$$k(\sigma) = h(s)$$

$$k(0_{\mathfrak{B},0}) = h(e)$$

$$k(0_{0,\mathfrak{B}}) = h(t)$$

If k is to be a functor, then

$$k(\Delta_i) = k(\sigma\delta_i) = k(\sigma)k(\delta_i) = h(s)h(d_i)$$

$$k(0_{0,\mathfrak{A}}) = k(\sigma 0_{0,\mathfrak{B}}) = k(\sigma)k(0_{0,\mathfrak{B}}) = h(s)h(t)$$

$$k(0_{\mathfrak{A},0}) = k(0_{\mathfrak{B},0}\delta_0) = k(0_{\mathfrak{B},0})k(\delta_0) = h(e)h(d_0)$$

$$k(0_{\mathfrak{A},\mathfrak{B}}) = k(0_{0,\mathfrak{B}})k(0_{\mathfrak{A},0}) = h(t)h(e)h(d_0)$$

$$k(0_{\mathfrak{B},\mathfrak{A}}) = k(0_{0,\mathfrak{A}})k(0_{\mathfrak{B},0}) = h(s)h(t)h(e)$$

$$k(0_{0,0}) = k(0_{\mathfrak{B},0})k(0_{0,\mathfrak{B}}) = h(e)h(t)$$

$$k(0_{\mathfrak{A},\mathfrak{A}}) = k(0_{0,\mathfrak{A}})k(0_{\mathfrak{A},0}) = h(s)h(t)h(e)h(d_0)$$

$$k(0_{\mathfrak{B},\mathfrak{B}}) = k(0_{0,\mathfrak{B}})k(0_{\mathfrak{B},0}) = h(t)h(e)$$

Using the fact that h coequalizes f_0, f_1, we can show that this indeed defines a functor. ■

So of course any split coequalizer is an arrow out of FG that coequalizes f_0, f_1, hence factors uniquely through r. So the split coequalizers in **Sets**, say, are the presheaves $\mathbf{Sets}^{\mathbf{Sc}} = \mathbf{Sets}^{(\mathbf{Sc}^{\mathrm{op}})^{\mathrm{op}}}$. The structure of the full subcategory of representable presheaves looks like the following: (4 stands for $\{0_{AA}, \Delta_1, \Delta_0, \mathrm{id}_A\}$ and 3 stands for $\{0_{AB}, \delta_1, \delta_0\}$)

$$\begin{tikzcd}
4 \arrow[r, shift left, "0122"] \arrow[r, shift right, "0001"'] & 3 \arrow[r, "!"] & 1 \\
2 \arrow[u, shift left, "02"] \arrow[u, shift right, "00"'] \arrow[r, equal, shift left] \arrow[r, shift right, "00"'] & 2 \arrow[u, shift left, "02"] \arrow[u, shift right, "00"'] \arrow[r, "!"] & 1 \arrow[u, equal] \\
1 \arrow[u, "0"] \arrow[r, equal, shift left] \arrow[r, equal, shift right] & 1 \arrow[u, "0"] \arrow[r, equal] & 1 \arrow[u, equal]
\end{tikzcd}$$

$$
\begin{tikzcd}
4 \arrow[d, "0011"'] & 3 \arrow[l, hook'] \arrow[d, "001" description] & 1 \arrow[l, hook'] \arrow[d, equal] \\
2 \arrow[d, "!"'] \arrow[r, equal] & 2 \arrow[d, "!"'] & 1 \arrow[l, hook'] \arrow[d, "!"] \\
1 \arrow[r, equal] & 1 \arrow[r, equal] & 1
\end{tikzcd}
$$