Notes on Labelling Linear Logic

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Abstract

One gives Kripke semantics for modal logic by identifying a class of worlds together with an accessibility relation between them and defining satisfaction at a world. One may give a semantics for bunched logic in a similar fashion by identifying the right notion of 'point relative to which satsifaction is defined'. This amounts intuitively to a set of resource labels which describe the resources divided up by the bunched logical connetives. Instead of (or, depending on circumstances, in addition to) a relational structure requires the *algebraic* structure of a commutative monoid, whose multiplication describes how resources combine.

Both of these semantics involve satisfaction clauses written against an assumed background of classical logic. To give a 'constructive semantics' is to give a notion of constructive proof of satisfaction that treats logical connectives' effect on those objects variously referred to as 'worlds', 'points', 'resource labels'. That is, a system of *labelled* deduction in the sense of Gabbay.

In that sense the contribution is a constructive resource semantics for intuitionistic linear logic without 0.

1 Introduction

I found it expedient to use yet another variant in the space of 'weak focalizations' of linear logic: here positive things are entirely unfocused, but negative things are fully focused, i.e. must continue focus when synchronous, and must be inverted eagerly. I believe everything goes fine if you fully focus the positives, too, but it did not seem necessary the way the negatives seemed to be. See the comment after the cut theorem statement in section 3.

Also, if the positives are required to do inversion, then we have some kind of substructrual context, whether ordered or 'linear but confluent', and that seems begging the question just a little to explain linearity in terms of linearity. Granted we're still using linearity in some capacity when talking about left focus, but since it's just one slot and not a whole context it seems more benign. That is, it can be accounted for by being a funny piece of syntax, without the generality of speaking of dividing up linear contexts and so on.

2 Negatively Focused Linear Logic

We take a polarized language with all the usual connectives of linear logic (excluding the exponential !, which is possible to treat without too much difficulty, but clutters the development somewhat) with polarity shifts $\uparrow$ and $\downarrow$ and signed atomic propositions a^+ and a^-.

$$\begin{array}{rccl}
\text{Positives} & P & ::= & \downarrow N \mid P \otimes P \mid P \oplus P \mid 1 \mid a^+ \\
\text{Negatives} & N & ::= & \uparrow P \mid P \multimap N \mid N \& N \mid \top \mid a^- \\
\text{Contexts} & \Delta & ::= & \cdot \mid \Delta, P \\
\text{Stable Conclusions} & Q & ::= & P \mid a^- \\
\text{Conclusions} & J & ::= & N > Q \mid P \mid N
\end{array}$$

Since we decompose negative propositions only in focus, the context Δ only contains positives. For the sake of writing general enough conclusions for sequent left rules, a stable conclusion Q is defined to be a positive proposition or a negative atom. The possible conclusions of a judgment are a negative proposition in focus with a stable conclusion $N > Q$, a negative proposition N under inversion, a positive proposition P, or a negative atom a^-. So the single judgment form is

$$\boxed{\Delta \vdash J}$$

with Δ unordered, but linear, i.e. without any tacit weakening and contraction.

The sequent calculus is

$$\frac{}{\vdash a^- > a^-} \qquad \frac{}{a^+ \vdash a^+}$$

$$\frac{\Delta_1 \vdash P_1 \qquad \Delta_2 \vdash P_2}{\Delta_1, \Delta_2 \vdash P_1 \otimes P_2} \qquad \frac{\Delta, P_1, P_2 \vdash Q}{\Delta, P_1 \otimes P_2 \vdash Q}$$

$$\frac{\Delta \vdash P_i}{\Delta \vdash P_1 \oplus P_2} \qquad \frac{\Delta, P_1 \vdash Q \qquad \Delta, P_2 \vdash Q}{\Delta, P_1 \oplus P_2 \vdash Q}$$

$$\frac{}{\vdash 1} \qquad \frac{\Delta \vdash Q}{\Delta, 1 \vdash Q} \qquad \frac{}{\Delta \vdash \top}$$

$$\frac{\Delta \vdash N_1 \qquad \Delta \vdash N_2}{\Delta \vdash N_1 \& N_2} \qquad \frac{\Delta \vdash N_i > Q}{\Delta \vdash N_1 \& N_2 > Q}$$

$$\frac{\Delta, P \vdash N}{\Delta \vdash P \multimap N} \qquad \frac{\Delta_1 \vdash P \qquad \Delta_2 \vdash N > Q}{\Delta_1, \Delta_2 \vdash P \multimap N > Q}$$

$$\frac{\Delta \vdash P}{\Delta \vdash \uparrow P} \qquad \frac{\Delta, P \vdash Q}{\Delta \vdash \uparrow P > Q}$$

$$\frac{\Delta \vdash N}{\Delta \vdash \downarrow N} \qquad \frac{\Delta \vdash N > Q}{\Delta, \downarrow N \vdash Q}$$

The soundness and completeness of the logic are expressed by

Theorem 2.1 (Cut) *The following rules are admissible.*

$$\frac{\Delta_1 \vdash P \qquad \Delta_2, P \vdash J}{\Delta_1, \Delta_2 \vdash J}$$

$$\frac{\Delta_1 \vdash N \qquad \Delta_2 \vdash N > Q}{\Delta_1, \Delta_2 \vdash Q}$$

$$\frac{\Delta_1 \vdash N > P \qquad \Delta_2, P \vdash Q}{\Delta_1, \Delta_2 \vdash N > Q}$$

Proof By lexicographic induction on the proposition and the structure of the derivations. ■

Theorem 2.2 (Identity)

1. *For any P, we have $P \vdash P$.*
2. *For any N, if there is a is a derivation uniform in Q from $\vdash N > Q$ to $\Delta \vdash Q$, then we have $\Delta \vdash N$*

Proof By induction on the structure of the proposition. ■

Corollary 2.3 *For any N we have $\downarrow N \vdash N$.*

Proof The left rule for $\downarrow$ allows a derivation

$$\frac{\vdash N > Q}{\downarrow N \vdash Q}$$

of the appropriate form. ■

3 Negatively Focused Labelled Logic

In the labelled logic we take the same language of propositions, but the contexts for it will admit weakening and contraction, as well as carry extra labelling information.

Labels	p, q, r	$::=$	$\alpha \mid p * p \mid \epsilon$
Labelled Contexts	Γ	$::=$	$\cdot \mid \Gamma, P[\alpha] \mid \Gamma, \alpha : \ell$
Labelled Conclusions	K	$::=$	$N[p] > Q[q] \mid P[p] \mid N[p]$

The label ϵ is to be understood as the empty collection of resources, and $*$ combines collections of resources. An appearance of a label variable α in a context as $P[\alpha]$ is a binding position, as is a direct declaration of a label variable $\alpha : \ell$. Everywhere else that α appears is a variable use. In this way, each positive hypothesis is marked as a distinct resource.

The main judgment form is

$$\boxed{\Gamma \vdash K}$$

and we have an auxiliary notion of well-formed label, $\Gamma \vdash : \ell$. It is defined by

$$\frac{}{\Gamma, \alpha : \ell \vdash \alpha : \ell} \qquad \frac{}{\Gamma \vdash \epsilon : \ell} \qquad \frac{\Gamma \vdash p : \ell \qquad \Gamma \vdash q : \ell}{\Gamma \vdash p * q : \ell}$$

The relation $p \equiv_{ACU} q$ is the equivalence (i.e. symmetric, reflexive, transitive) closure of **A**ssociativity, **C**ommutativity and **U**nit laws for $*$ with respect to ϵ, and compatibility with respect to $*$, i.e. $p * q \equiv_{ACU} p' * q'$ if $p \equiv_{ACU} p'$ and $q \equiv_{ACU} q'$.

In the sequel we identify label expressions p and q that satisfy $p \equiv_{ACU} q$.

Substitution of a label p for a label variable α in q is written $(p/\alpha)q$.

The rules defining the sequent calculus are

$$\frac{}{\Gamma \vdash a^-[p] > a^-[p]} \qquad \frac{}{\Gamma, a^+[\alpha] \vdash a^+[\alpha]}$$

$$\frac{\Gamma \vdash P_1[p_1] \qquad \Gamma \vdash P_2[p_2]}{\Gamma \vdash P_1 \otimes P_2[p_1 * p_2]} \qquad \frac{\Gamma, P_1 \otimes P_2[\alpha], P_1[\beta_1], P_2[\beta_2] \vdash Q[(\beta_1 * \beta_2/\alpha)p]}{\Gamma, P_1 \otimes P_2[\alpha] \vdash Q[p]}$$

$$\frac{\Gamma \vdash P_i[p]}{\Gamma \vdash P_1 \oplus P_2[p]} \qquad \frac{\Gamma, P_1[\alpha] \vdash Q[q] \qquad \Gamma, P_2[\alpha] \vdash Q[q]}{\Gamma, P_1 \oplus P_2[\alpha] \vdash Q[q]}$$

$$\frac{}{\vdash 1[\epsilon]} \qquad \frac{\Gamma \vdash Q[(\epsilon/\alpha)p]}{\Gamma, 1[\alpha] \vdash Q[p]} \qquad \frac{}{\Gamma \vdash \top[p]}$$

$$\frac{\Gamma \vdash N_1[p] \qquad \Gamma \vdash N_2[p]}{\Gamma \vdash N_1 \& N_2[p]} \qquad \frac{\Gamma \vdash N_i[p] > Q[q]}{\Gamma \vdash N_1 \& N_2[p] > Q[q]}$$

$$\frac{\Gamma, P[\alpha] \vdash N[p * \alpha]}{\Gamma \vdash P \multimap N[p]} \qquad \frac{\Gamma \vdash P[r] \qquad \Gamma \vdash N[p * r] > Q[q]}{\Gamma \vdash P \multimap N[p] > Q[q]}$$

$$\frac{\Gamma \vdash P[p]}{\Gamma \vdash \uparrow P[p]} \qquad \frac{\Gamma, P[\alpha] \vdash Q[q]}{\Gamma \vdash \uparrow P[p] > Q[(p/\alpha)q]}$$

$$\frac{\Gamma \vdash N[p]}{\Gamma \vdash \downarrow N[p]} \qquad \frac{\Gamma, \downarrow N[\alpha], \beta : \ell \vdash N[\beta] > Q[(\beta/\alpha)q]}{\Gamma, \downarrow N[\alpha] \vdash Q[q]}$$

The soundness and completeness of the logic are expressed by

Theorem 3.1 (Cut) *The following rules are admissible.*

$$\frac{\Gamma \vdash p : \ell \qquad \Gamma, \alpha : \ell \vdash K}{\Gamma \vdash (p/\alpha)K} \qquad \frac{\Gamma \vdash P[p] \qquad \Gamma, P[\alpha] \vdash K}{\Gamma \vdash (p/\alpha)K}$$

$$\frac{\Gamma \vdash N[p] \qquad \Gamma \vdash N[p] > Q[q]}{\Gamma \vdash Q[q]} \qquad \frac{\Gamma \vdash N[p] > P[q] \qquad \Gamma, P[\alpha] \vdash Q[r]}{\Gamma \vdash N[p] > Q[(q/\alpha)r]}$$

Proof By lexicographic induction on the proposition and the structure of the derivations. $\blacksquare$

The success of this proof approach depends on the imposition of eager inversion for negative propositions on the right. Otherwise, it would not be obvious how to maintain the invariant that negative propositions are cut against negative propositions at the same world. Consider the commutative case (which uses a conjectural variant of the $\downarrow$ left rule, critically not actually allowed in the system above)

$$\frac{\dfrac{\Gamma, P[b] \vdash N[p]}{\Gamma, \downarrow P[r] \vdash N[(r/\beta)p]} \qquad \Gamma, \downarrow P[r] \vdash N[(r/\beta)p] > Q[q]}{\Gamma, \downarrow P[r] \vdash Q[q]}$$

The reason for left focus is more basic — it is the place where hypotheses at complex labels (not single variables) makes sense.

Theorem 3.2 (Identity)

1. *For any P, we have $P[\alpha] \vdash P[\alpha]$.*
2. *For any N, if there is a is a derivation uniform in Q from $\beta : \ell \vdash N[\beta] > Q[(\beta/\alpha)p]$ to $\Gamma \vdash Q[p]$, then we have $\Gamma \vdash N[\alpha]$ XXX not quite right yet*

Proof By induction on the structure of the proposition. $\blacksquare$

Corollary 3.3 *For any N we have $\downarrow N[\alpha] \vdash N[\alpha]$.*

Proof The left rule for $\downarrow$ allows a derivation

$$\frac{\downarrow N[\alpha], \alpha : \ell \vdash N[\alpha] > Q[p]}{\downarrow N[\alpha] \vdash Q[p]}$$

of the appropriate form. $\blacksquare$

4 Embedding

The labelled logic conservatively extends linear logic in the sense of the two results below. First some preliminary definitions. The expression $\Gamma\big|_p$ for a p consisting of a product of a subset of the world variables declared by positive hypotheses in Γ is defined to be the linear context arising from taking precisely those positive hypotheses. Formally,

$$\frac{\Gamma\big|_p = \Delta}{(\Gamma, P[\alpha])\big|_{p*\alpha} = \Delta, P} \qquad \frac{}{\Gamma\big|_\epsilon = \cdot}$$

Besides that definition it's useful to be able to say by itself that p is actually a subset of the variables in Γ, (that is, it does not duplicate any of them) written $p \subseteq \Gamma$

$$\frac{p \subseteq \Gamma}{p * \alpha \subseteq \Gamma, \alpha : \ell} \qquad \frac{p \subseteq \Gamma}{p * \alpha \subseteq \Gamma, P[\alpha]} \qquad \frac{}{\epsilon \subseteq \Gamma}$$

The intent then is that $\Gamma\big|_p$ should only be invoked if it's already known that $p \subseteq \Gamma$.

If we have a linear context $\Delta = P_1 \dots P_n$ then α_Δ is a product of fresh label variables of the same length as Δ, i.e. $\alpha_1 * \cdots * \alpha_n$ and $\Delta[\alpha_\Delta]$ is defined to be $P_1[\alpha_1] \dots P_n[\alpha_n]$.

Theorem 4.1 (Soundness)

1. *If $p \subseteq \Gamma$ and $\Gamma \vdash P[p]$ then $\Gamma\big|_p \vdash P$.*

2. *If $p \subseteq \Gamma$ and $\Gamma \vdash N[p]$ then $\Gamma\big|_p \vdash N$.*

3. *If $p \subseteq \Gamma$ and $\Gamma \vdash N[q] > Q[p]$ then there is an r such that $p \equiv_{ACU} q * r$ and $\Gamma\big|_r \vdash N > Q$.*

Proof By induction on the derivation. XXX I cannot quite see yet how to do the cases of 3 for the shifts, but everything else looks good. ■

Theorem 4.2 (Completeness) *Let Δ be given and let Γ be a context extending $\Delta[\alpha_\Delta]$. Suppose $\Gamma \vdash p : \ell$.*

1. *If $\Delta \vdash P$ then $\Gamma \vdash P[\alpha_\Delta]$.*

2. *If $\Delta \vdash N$ then $\Gamma \vdash N[\alpha_\Delta]$.*

3. *If $\Delta \vdash N > Q$ then $\Gamma \vdash N[p] > Q[p * \alpha_\Delta]$.*

Proof By induction on the derivation. ■

4.1 Additive Falsehood 0

What to do about 0 is not clear. Taking it to be defined in the two systems by

$$\frac{}{\Delta, 0 \vdash Q} \qquad \frac{}{\Gamma, 0[\alpha] \vdash Q[p]}$$

breaks soundness, because you can then prove $\downarrow(a^- \& (a^- \multimap \uparrow 0)) \vdash 0$ in the labelled system and not in linear logic. Yet this seems to be the correct right rule for 0 in the labelled system, and not for instance something like

$$\frac{}{\Gamma, 0[\alpha] \vdash Q[\alpha * p]}$$

which would at least naïvely rule out using 0 when it's not currently 'really in the linear context'.