LF with Irrelevance and Notational Definitions

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Contents

1 Language

Kinds	K	$::= \mathsf{type} \mid \Pi x{:}A.K \mid \Pi x{\div}A.K$
Types	A	$::= a \cdot S \mid \Pi x{:}A.B \mid \Pi x{\div}A.B$
Objects	M	$::= H \cdot S \mid \lambda x{:}A.M \mid \lambda x{\div}A.M \mid M \cdot S$
Spines	S	$::= \mathsf{nil} \mid (M; S) \mid (M \div S)$
Heads	H	$::= c \mid x \mid d$
Signatures	Σ	$::= \cdot \mid \Sigma, a : K \mid \Sigma, c : A \mid \Sigma, d : A = M$
Contexts	Γ	$::= \cdot \mid \Gamma, x : A \mid \Gamma, x \div A$
Erased Contexts	$\Gamma^{\Box}$	$::= \cdot \mid \Gamma^{\Box}, x$

2 Proof Typing

Functions:

Proof Promotion	$\Gamma \mapsto \Gamma^{\oplus}$	Promote proof variables in Γ to term variables

Judgments:

Proof Object Typing	$\Gamma \vdash M \div A$	In Γ, M has irrelevant type A.

2.1 Proof Promotion

$$\begin{array}{rcl} \cdot^{\oplus} & := & \cdot \\ (\Gamma, x : A)^{\oplus} & := & \Gamma^{\oplus}, x : A \\ (\Gamma, x \div A)^{\oplus} & := & \Gamma^{\oplus}, x : A \end{array}$$

2.2 Proof Object Typing

$$\frac{\Gamma^{\oplus} \vdash M : A \qquad \Gamma \vdash A : \mathsf{type}}{\Gamma \vdash M \div A}\,\mathsf{pot}$$

3 Typing

Judgments:

Spine Typing	$\Gamma \vdash S : A > a$	In Γ, S gives base type a for a head of type A.
Type Spine Kinding	$\Gamma \vdash S : K \gg \mathsf{type}$	In Γ, S gives a type for a head of kind K.
Object Typing	$\Gamma \vdash M : A$	In Γ, M has type A.
Type Validity	$\Gamma \vdash A : \mathsf{type}$	In Γ, A is a valid type.
Kind Validity	$\Gamma \vdash K : \mathsf{kind}$	In Γ, K is a valid kind.

3.1 Spine Typing

$$\frac{}{\Gamma \vdash \mathsf{nil} : A > A}\,\mathsf{st\_nil}$$

$$\frac{\Gamma \vdash M : A \qquad \Gamma \vdash S : [M/x]B > C}{\Gamma \vdash (M; S) : \Pi x{:}A.B > C}\,\mathsf{st\_cons}$$

$$\frac{\Gamma \vdash M \div A \qquad \Gamma \vdash S : [M/x]B > C}{\Gamma \vdash (M \div S) : \Pi x{\div}A.B > C}\,\mathsf{st\_icons}$$

$$\frac{\Gamma \vdash S : A > B \qquad \Gamma \vdash B = B' : \mathsf{type}}{\Gamma \vdash S : A > B'}\,\mathsf{st\_conv}$$

3.2 Type Spine Kinding

$$\frac{}{\Gamma \vdash \mathsf{nil} : \mathsf{type} \gg \mathsf{type}}\,\mathsf{tsk\_nil}$$

$$\frac{\Gamma \vdash M : A \qquad \Gamma \vdash S : [M/x]K \gg \mathsf{type}}{\Gamma \vdash (M; S) : \Pi x{:}A.K \gg \mathsf{type}}\,\mathsf{tsk\_cons}$$

$$\frac{\Gamma \vdash M \div A \qquad \Gamma \vdash S : [M/x]K \gg \mathsf{type}}{\Gamma \vdash (M \div S) : \Pi x{\div}A.K \gg \mathsf{type}}\,\mathsf{tsk\_icons}$$

3.3 Object Typing

$$\frac{c : A \in \Sigma \qquad \Gamma \vdash S : A > B}{\Gamma \vdash c \cdot S : B}\,\mathsf{ot\_const}$$

$$\frac{x : A \in \Gamma \qquad \Gamma \vdash S : A > B}{\Gamma \vdash x \cdot S : B}\,\mathsf{ot\_var}$$

$$\frac{d : A = M \in \Sigma \qquad \Gamma \vdash S : A > B}{\Gamma \vdash d \cdot S : B}\,\mathsf{ot\_def}$$

$$\frac{\Gamma, x \star A \vdash M : B \qquad \Gamma \vdash A : \mathsf{type}}{\Gamma \vdash \lambda x{\star}A.M : \Pi x{\star}A.B}\,\mathsf{ot\_lam}$$

$$\frac{\Gamma \vdash M : A \qquad \Gamma \vdash S : A > B}{\Gamma \vdash M \cdot S : B}\,\textsf{ot\_app}$$

$$\frac{\Gamma \vdash M : A \qquad \Gamma \vdash A = B : \textsf{type}}{\Gamma \vdash M : B}\,\textsf{ot\_conv}$$

3.4 Type Validity

$$\frac{a : K \in \Sigma \qquad \Gamma \vdash S : K \gg \textsf{type}}{\Gamma \vdash a \cdot S : \textsf{type}}\,\textsf{tv\_const}$$

$$\frac{\Gamma \vdash A : \textsf{type} \qquad \Gamma, x \star A \vdash B : \textsf{type}}{\Gamma \vdash \Pi x{\star}A.B : \textsf{type}}\,\textsf{tv\_pi}$$

3.5 Kind Validity

$$\frac{}{\Gamma \vdash \textsf{type} : \textsf{kind}}\,\textsf{kv\_type}$$

$$\frac{\Gamma \vdash A : \textsf{type} \qquad \Gamma, x \star A \vdash K : \textsf{kind}}{\Gamma \vdash \Pi x{\star}A.K : \textsf{kind}}\,\textsf{kv\_pi}$$

4 Equality

Judgments:

Proof Equality	$\Gamma \vdash M_1 = M_2 \div A$	In Γ, $M_1 = M_2$ at irrelevant type A.
Object Equality	$\Gamma \vdash M_1 = M_2 : A$	In Γ, $M_1 = M_2$ at type A.
Type Equality	$\Gamma \vdash A_1 = A_2 : \textsf{type}$	In Γ, A_1 and A_2 are equal types.
Kind Equality	$\Gamma \vdash K_1 = K_2 : \textsf{kind}$	In Γ, K_1 and K_2 are equal kinds.
Spine Equality	$\Gamma \vdash S_1 = S_2 : A > B$	In Γ, $S_1 = S_2$ at type $A > B$.
Type Spine Equality	$\Gamma \vdash S_1 = S_2 : K \gg \textsf{type}$	In Γ, $S_1 = S_2$ at kind $\textsf{type}$.

4.1 Proof Equality

$$\frac{\Gamma^{\oplus} \vdash M = M : A \qquad \Gamma^{\oplus} \vdash N = N : A \qquad \Gamma \vdash A : \textsf{type}}{\Gamma \vdash M = N \div A}\,\textsf{pe}$$

4.2 Object Equality

4.2.1 Simultaneous Congruence

$$\frac{c : A \in \Sigma \qquad \Gamma \vdash S_1 = S_2 : A > B}{\Gamma \vdash c \cdot S_1 = c \cdot S_2 : B}\,\textsf{oe\_const}$$

$$\frac{x : A \in \Gamma \qquad \Gamma \vdash S_1 = S_2 : A > B}{\Gamma \vdash x \cdot S_1 = x \cdot S_2 : B}\,\textsf{oe\_var}$$

$$\frac{d : A = M \in \Sigma \qquad \Gamma \vdash M \cdot S_1 = M \cdot S_2 : B}{\Gamma \vdash d \cdot S_1 = d \cdot S_2 : B}\,\textsf{oe\_def}$$

$$\frac{\Gamma \vdash A_1' = A_1 : \textsf{type} \qquad \Gamma \vdash A_1'' = A_1 : \textsf{type} \qquad \Gamma, x \star A_1 \vdash M = N : A_2}{\Gamma \vdash \lambda x{\star}A_1'.M = \lambda x{\star}A_1''.N : \Pi x{\star}A_1.A_2}\,\textsf{oe\_lam}$$

$$\frac{\Gamma \vdash M_1 = M_2 : A \qquad \Gamma \vdash S_1 = S_2 : A > B}{\Gamma \vdash M_1 \cdot S_1 = M_2 \cdot S_2 : B}\,\textsf{oe\_app}$$

4.2.2 Extensionality

$$\frac{\Gamma \vdash A_1 : \textsf{type} \qquad \begin{array}{c}\{\Gamma \vdash M_1, M_2 : \Pi x{:}A_1.A_2\} \\ \Gamma, x : A_1 \vdash M_1 \cdot (x \cdot \textsf{nil}; \textsf{nil}) = M_2 \cdot (x \cdot \textsf{nil}; \textsf{nil}) : A_2\end{array}}{\Gamma \vdash M_1 = M_2 : \Pi x{:}A_1.A_2}\,\textsf{oe\_ext}$$

$$\frac{\Gamma \vdash A_1 : \textsf{type} \qquad \begin{array}{c}\{\Gamma \vdash M_1, M_2 : \Pi x{\div}A_1.A_2\} \\ \Gamma, x \div A_1 \vdash M_1 \cdot (x \cdot \textsf{nil} \,\dot{;}\, \textsf{nil}) = M_2 \cdot (x \cdot \textsf{nil} \,\dot{;}\, \textsf{nil}) : A_2\end{array}}{\Gamma \vdash M_1 = M_2 : \Pi x{\div}A_1.A_2}\,\textsf{oe\_iext}$$

4.2.3 Parallel Reduction

$$\frac{\Gamma \vdash M = N : B}{\Gamma \vdash M \cdot \textsf{nil} = N : B}\,\textsf{oe\_red\_nil}$$

$$\frac{\Gamma \vdash H \cdot \textsf{nil} : B \qquad \Gamma \vdash S_1 = S_2 : B > C}{\Gamma \vdash (H \cdot \textsf{nil}) \cdot S_1 = H \cdot S_2 : C}\,\textsf{oe\_red\_nil2}$$

$$\frac{\begin{array}{l}\Gamma, x : B \vdash M_1 = M_2 : A \\ \qquad \Gamma \vdash N_1 = N_2 : B \qquad \Gamma, x : B \vdash S = S' : A > C\end{array}}{\Gamma \vdash (\lambda x{:}B.M_1) \cdot (N_1; S) = [N_2/x]M_2 \cdot S' : [N_1/x]C}\,\textsf{oe\_red\_cons}$$

$$\frac{\begin{array}{l}\Gamma, x \div B \vdash M_1 = M_2 : A \\ \qquad \Gamma \vdash N_1 = N_2 \div B \qquad \Gamma, x \div B \vdash S = S' : A > C\end{array}}{\Gamma \vdash (\lambda x{\div}B.M_1) \cdot (N_1 \,\dot{;}\, S) = [N_2/x]M_2 \cdot S' : [N_1/x]C}\,\textsf{oe\_red\_icons}$$

4.2.4 Type Conversion

$$\frac{\Gamma \vdash M = N : A \qquad \Gamma \vdash A = B : \mathsf{type}}{\Gamma \vdash M = N : B}\ \mathsf{oe\_conv}$$

4.3 Type Equality

4.3.1 Simultaneous Congruence

$$\frac{a : K \in \Sigma \qquad \Gamma \vdash S_1 = S_2 : K \gg \mathsf{type}}{\Gamma \vdash a \cdot S_1 = a \cdot S_2 : \mathsf{type}}\ \mathsf{te\_const}$$

$$\frac{\Gamma \vdash A' = A : \mathsf{type} \qquad \Gamma, x \star A \vdash B = B' : \mathsf{type}}{\Gamma \vdash \Pi x {\star} A.B = \Pi x {\star} A'.B' : \mathsf{type}}\ \mathsf{te\_pi}$$

4.4 Kind Equality

4.4.1 Simultaneous Congruence

$$\frac{}{\Gamma \vdash \mathsf{type} = \mathsf{type} : \mathsf{kind}}\ \mathsf{ke\_type}$$

$$\frac{\Gamma \vdash A' = A : \mathsf{type} \qquad \Gamma, x \star A \vdash K = K' : \mathsf{kind}}{\Gamma \vdash \Pi x {\star} A.K = \Pi x {\star} A'.K' : \mathsf{kind}}\ \mathsf{ke\_pi}$$

4.5 Spine Equality

4.5.1 Simultaneous Congruence

$$\frac{}{\Gamma \vdash \mathsf{nil} = \mathsf{nil} : A > A}\ \mathsf{se\_nil}$$

$$\frac{\Gamma \vdash M_1 = M_2 : A \qquad \Gamma \vdash S_1 = S_2 : [M_1/x]B > C}{\Gamma \vdash (M_1; S_1) = (M_2; S_2) : \Pi x{:}A.B > C}\ \mathsf{se\_cons}$$

$$\frac{\Gamma \vdash M_1 = M_2 \div A \qquad \Gamma \vdash S_1 = S_2 : [M_1/x]B > C}{\Gamma \vdash (M_1 \,\dot{;}\, S_1) = (M_2 \,\dot{;}\, S_2) : \Pi x {\div} A.B > C}\ \mathsf{se\_icons}$$

4.6 Type Spine Equality

4.6.1 Simultaneous Congruence

$$\frac{}{\Gamma \vdash \mathsf{nil} = \mathsf{nil} : \mathsf{type} \gg \mathsf{type}}\ \mathsf{tse\_nil}$$

$$\frac{\Gamma \vdash M_1 = M_2 : A \qquad \Gamma \vdash S_1 = S_2 : [M_1/x]K \gg \mathsf{type}}{\Gamma \vdash (M_1; S_1) = (M_2; S_2) : \Pi x{:}A.K \gg \mathsf{type}}\ \mathsf{tse\_cons}$$

$$\frac{\Gamma \vdash M_1 = M_2 \div A \qquad \Gamma \vdash S_1 = S_2 : [M_1/x]K \gg \mathsf{type}}{\Gamma \vdash (M_1 \,\dot{;}\, S_1) = (M_2 \,\dot{;}\, S_2) : \Pi x {\div} A.K \gg \mathsf{type}}\ \mathsf{tse\_icons}$$

Lemma 4.1 (d)

Proof ■

Lemma 4.2 *(η-conversion)*

1. *If* $\Gamma \vdash M : \Pi x{:}A.B$*, then* $\Gamma \vdash M = \lambda x{:}A.M \cdot (x \cdot \mathsf{nil}; \mathsf{nil}) : \Pi x{:}A.B$

2. *If* $\Gamma \vdash M : \Pi x{\div}A.B$*, then* $\Gamma \vdash M = \lambda x{\div}A.M \cdot (x \cdot \mathsf{nil} \div \mathsf{nil}) : \Pi x{\div}A.B$

Proof

1.

$$\dfrac{\dfrac{}{x \cdot \mathsf{nil} = (x \cdot \mathsf{nil}) \cdot \mathsf{nil}}\,\mathsf{oe\_red\_nil} \qquad \dfrac{}{(x \cdot \mathsf{nil}; \mathsf{nil}) = (x \cdot \mathsf{nil}; \mathsf{nil})}}{\dfrac{\Gamma, x : A \vdash M \cdot (x \cdot \mathsf{nil}; \mathsf{nil}) = (\lambda y{:}A.M \cdot (y \cdot \mathsf{nil}; \mathsf{nil})) \cdot (x \cdot \mathsf{nil}; \mathsf{nil}) : B}{\Gamma \vdash M = \lambda y{:}A.M \cdot (y \cdot \mathsf{nil}; \mathsf{nil}) : \Pi y{:}A.B}\,\mathsf{oe\_ext}}\,\mathsf{oe\_red\_cons}$$

2. Similar.

■

Lemma 4.3 *(Spine Splicing)*

1. *If* $\Gamma \vdash S : A > B$, $\Gamma \vdash M : \Pi x{:}C.A$ *and* $\Gamma \vdash N : C$*, then* $\Gamma \vdash M \cdot (N; S) = (M \cdot (N; \mathsf{nil})) \cdot S : B$

2. *If* $\Gamma \vdash S : A > B$, $\Gamma \vdash M : \Pi x{\div}C.A$ *and* $\Gamma \vdash N{\div}C$*, then* $\Gamma \vdash M \cdot (N \div S) = (M \cdot (N \div \mathsf{nil})) \cdot S : B$

Proof 1. By Lemma 4.1 and $\mathsf{oe\_app}$, we know

$$\Gamma \vdash M \cdot (N; S) = (\lambda x{:}A.M \cdot (x \cdot \mathsf{nil}; \mathsf{nil})) \cdot (N; S) : B$$

By $\mathsf{oe\_red\_cons}$, (using $\mathsf{oe\_red\_nil}$ to see that $N \cdot \mathsf{nil} = N$) we have

$$\Gamma \vdash M \cdot (N; S) = (M \cdot (N; \mathsf{nil})) \cdot S : B$$

as required.

2. Similar.

■

Lemma 4.4 *(Validity)*

1. *If* $\Gamma \vdash M \star A$*, then* $\Gamma \vdash A : \mathsf{type}$.

2. *If* $\Gamma \vdash S : A > B$ *and* $\Gamma \vdash A : \mathsf{type}$*, then* $\Gamma \vdash B : \mathsf{type}$.

3. *If* $\Gamma, x \star A \vdash B : \mathsf{type}$ *and* $\Gamma \vdash M \star A$*, then* $\Gamma \vdash [M/x]B : \mathsf{type}$.

4. *If* $\Gamma \vdash M_1 = M_2 \star A$*, then* $\Gamma \vdash M_1 \star A$ *and* $\Gamma \vdash M_2 \star A$.

5. *If* $\Gamma \vdash S_1 = S_2 : A > B$*, then* $\Gamma \vdash S_1 : A > B$ *and* $\Gamma \vdash S_2 : A > B$.

6. *If* $\Gamma \vdash A_1 = A_2 : \mathsf{type}$*, then* $\Gamma \vdash A_1 : \mathsf{type}$ *and* $\Gamma \vdash A_2 : \mathsf{type}$.

Proof By induction on the appropriate rules. ■

Lemma 4.5 *(Unicity)*

1. *If* $\Gamma \vdash M \star A$ *and* $\Gamma \vdash M \star A'$*, then* $\Gamma \vdash A = A' : \mathsf{type}$.

2. *If* $\Gamma \vdash S : A > B$*, and* $\Gamma \vdash S : A' > B'$*, and also* $\Gamma \vdash A = A' : \mathsf{type}$*, then* $\Gamma \vdash B = B' : \mathsf{type}$.

Proof By simultaneous induction on the structure of the relevant typing deductions. ■

5 Strictness

Functions:

Erasure	$\Gamma \mapsto \Gamma^{\Box}$	Remove types and irrelevant variables from Γ

Judgments:

Pattern Spine	$\Delta^{\Box} \vdash S\ \mathsf{pat}$	In Δ, S is a pattern spine.
Local Strict Occurrences	$\Gamma; \Delta \Vdash_x M$	x occurs locally strict in M
	$\Gamma; \Delta \Vdash_x A$	x occurs locally strict in A
	$\Gamma; \Delta \Vdash_x S$	x occurs locally strict in S
Strict Occurrences	$\Gamma \Vdash_x M$	x occurs strict in M
Strictness	$\Gamma \Vdash M$	M is strict

5.1 Erasure

$$\begin{array}{rcl} \cdot^{\Box} & := & \cdot \\ (\Gamma, x : A)^{\Box} & := & \Gamma^{\Box}, x^{:} \\ (\Gamma, x \div A)^{\Box} & := & \Gamma^{\Box}, x^{\div} \end{array}$$

5.2 Pattern Spines

$$\frac{}{\Delta^{\Box} \vdash \mathsf{nil}\ \mathsf{pat}}\ \mathsf{pat\_nil}$$

$$\frac{\Delta^{\Box} \vdash S\ \mathsf{pat} \qquad x^{\star} \notin \Delta^{\Box}}{\Delta^{\Box}, x^{:} \vdash (x; S)\ \mathsf{pat}}\ \mathsf{pat\_cons}$$

$$\frac{\Delta^{\Box} \vdash S\ \mathsf{pat}}{\Delta^{\Box} \vdash (M \,\dot{;}\, S)\ \mathsf{pat}}\ \mathsf{pat\_icons}$$

Lemma 5.1 *Suppose* $\Gamma \vdash N_1, N_2 \div A$.

1. *If* $\Gamma, z \div A \vdash M \star B$, *then* $\Gamma \vdash [N_1/z]M = [N_2/z]M \star [N_1/z]B$.
2. *If* $\Gamma, z \div A \vdash S : A > B$, *then* $\Gamma \vdash [N_1/z]S = [N_2/z]S : [N_1/z]A > [N_1/z]B$.
3. *If* $\Gamma, z \div A \vdash B \star \mathsf{type}$, *then* $\Gamma \vdash [N_1/z]B = [N_2/z]B \star \mathsf{type}$.

Proof Simultaneous induction on the object and spine typing and type validity rules. ■

Lemma 5.2

1. *If* $\Gamma \vdash M : A$, *then* $\Gamma \vdash [x \cdot \mathsf{nil}/x]M = M : A$.
2. *If* $\Gamma \vdash S : A > B$, *then* $\Gamma \vdash [x \cdot \mathsf{nil}/x]S = S : A > B$.
3. *If* $\Gamma \vdash A : \mathsf{type}$, *then* $\Gamma \vdash [x \cdot \mathsf{nil}/x]A = A : \mathsf{type}$.

Proof Simultaneous induction on the object and spine typing and type validity rules. The critical case is ot_var, where we use oe_red_nil2. ■

Lemma 5.3 *If* $\Gamma \vdash \Pi y{\star}B.B' : \mathsf{type}$ *and* $x \notin \Gamma$, *then* $\Gamma \vdash \Pi x{\star}B.[x \cdot \mathsf{nil}/y]B' = \Pi y{\star}B.B' : \mathsf{type}$.

Proof By α-conversion, we know $\Gamma \vdash \Pi x{\star}B.[x/y]B' : \mathsf{type}$, and inversion on the type validity rules gives $\Gamma, x \star B \vdash [x/y]B' : \mathsf{type}$. By Lemma 5.2 and the fact that $x \notin \Gamma$ and consequently x does not appear free in B', we know $\Gamma, x \star B \vdash [x \cdot \mathsf{nil}/y]B' : \mathsf{type}$, hence $\Gamma \vdash \Pi x{\star}B.[x \cdot \mathsf{nil}/y]B' : \mathsf{type}$, as required. ■

Lemma 5.4 *If*

1. $\Delta^{\Box} \vdash S\ \mathsf{pat}$,
2. $\Gamma \vdash M_1, M_2 : A_2$,
3. $\Delta \vdash S : A_1 > C$,
4. $\Gamma, \Delta \vdash A_1 = A_2 : \mathsf{type}$
5. $\Gamma, \Delta \vdash M_1 \cdot S = M_2 \cdot S : C$, *and*
6. *The variable names in* Γ *and* Δ *are disjoint.*

then $\Gamma \vdash M_1 = M_2 : A_2$.

Proof By induction on $\mathcal{D} :: \Delta^{\Box} \vdash S\ \mathsf{pat}$.

Case:

$$\mathcal{D} = \frac{}{\Delta^{\square} \vdash \mathsf{nil\ pat}}\ \mathsf{pat\_nil}$$

In this case, $S = \mathsf{nil}$. By inversion on the spine typing rules, A_1 is identical to C, so we have $\Gamma, \Delta \vdash C = A_2 : \mathsf{type}$. We know that $\Gamma, \Delta \vdash M_1 \cdot \mathsf{nil} = M_2 \cdot \mathsf{nil} : C$, so by $\mathsf{oe\_red\_nil}$ we can conclude $\Gamma, \Delta \vdash M_1 = M_2 : C$. By $\mathsf{oe\_conv}$, we know $\Gamma, \Delta \vdash M_1 = M_2 : A_2$. But no variable from Δ appears free in M_1 or M_2, so $\Gamma \vdash M_1 = M_2 : A_2$.

Case:

$$\mathcal{D} = \frac{\begin{array}{c}\mathcal{D}' \\ (\Delta')^{\square} \vdash S'\ \mathsf{pat}\end{array} \quad x^{\star} \notin (\Delta')^{\square}}{(\Delta')^{\square}, x^{:} \vdash (x \cdot \mathsf{nil}; S')\ \mathsf{pat}}\ \mathsf{pat\_cons}$$

In this case, $S = (x \cdot \mathsf{nil}; S')$ and $x : B \in \Delta$ for some $B : \mathsf{type}$. So we know $\Delta \vdash (x \cdot \mathsf{nil}; S') : A_1 > C$. Assumption (5) in this case is

$$\Gamma, \Delta \vdash M_1 \cdot (x \cdot \mathsf{nil}; S') = M_2 \cdot (x \cdot \mathsf{nil}; S') : C$$

By Lemma 4.2 we can infer

$$\Gamma, \Delta \vdash (M_1 \cdot (x \cdot \mathsf{nil}; \mathsf{nil})) \cdot S' = (M_2 \cdot (x \cdot \mathsf{nil}; \mathsf{nil})) \cdot S' : C$$

Now by inversion on the spine typing rules, A_1 must be of the form $\Pi y{:}B.B'$, such that $\Delta' \vdash S' : [x \cdot \mathsf{nil}/y]B' > C$. Notice that we can also now show that

$$\Gamma, x : B \vdash M_1 \cdot (x \cdot \mathsf{nil}; \mathsf{nil}), M_2 \cdot (x \cdot \mathsf{nil}; \mathsf{nil}) : [x \cdot \mathsf{nil}/y]B'$$

and $\Gamma, \Delta \vdash B' = B' : \mathsf{type}$, and also $\Gamma, x : B$ is disjoint from Δ'. By the induction hypothesis, we get

$$\Gamma, x : B \vdash (M_1 \cdot (x \cdot \mathsf{nil}; \mathsf{nil})) = (M_2 \cdot (x \cdot \mathsf{nil}; \mathsf{nil})) : [x \cdot \mathsf{nil}/y]B'$$

Using $\mathsf{oe\_ext}$, and Lemma 5.3, we conclude

$$\Gamma, \Delta \vdash M_1 = M_2 : \Pi y{:}B.B'$$

and so

$$\Gamma \vdash M_1 = M_2 : A_2$$

by type conversion and the observation that no variables in Δ appear free in M_1 or M_2 by their typing.

Case:

$$\mathcal{D} = \frac{\begin{array}{c}\mathcal{D}' \\ \Delta^{\square} \vdash S'\ \mathsf{pat}\end{array}}{\Delta^{\square} \vdash (M' \div S')\ \mathsf{pat}}\ \mathsf{pat\_icons}$$

In this case, $S = (M' \div S')$. By inversion on the spine typing rules, $\Delta \vdash M' \div B$ for some $B \div \mathsf{type}$ and A_1 is of the form $\Pi y {\div} B.B'$, such that $\Delta \vdash S' : [M'/y]B' > C$. So we know $\Delta \vdash (M' \div S') : \Pi y {\div} B.B' > C$. Assumption (5) in this case is

$$\Gamma, \Delta \vdash M_1 \cdot (M' \div S') = M_2 \cdot (M' \div S') : C$$

By Lemma 4.2 we can infer

$$\Gamma, \Delta \vdash (M_1 \cdot (M' \div \mathsf{nil})) \cdot S' = (M_2 \cdot (M' \div \mathsf{nil})) \cdot S' : C$$

Now create a fresh variable $\bar{x} : A$ for each $x : A \in \Delta$. Put $\bar{\Delta} = \{\bar{x} : A \mid x : A \in \Delta\}$. We have (by the triviality of equality at irrelevant type) that

$$\Delta, \bar{\Delta} \vdash M' = [\bar{\Delta}/\Delta]M' \div B$$

and so

$$\Gamma, \bar{\Delta}, \Delta \vdash (M_1 \cdot ([\bar{\Delta}/\Delta]M' \div \mathsf{nil})) \cdot S' = (M_2 \cdot ([\bar{\Delta}/\Delta]M' \div \mathsf{nil})) \cdot S' : C$$

by using Lemma 5.1. Notice that we do not substitute for any occurrences of M' in S'. We can also now show that

$$\Gamma, \bar{\Delta} \vdash M_1 \cdot ([\bar{\Delta}/\Delta]M' \div \mathsf{nil}), M_2 \cdot ([\bar{\Delta}/\Delta]M' \div \mathsf{nil}) : [[\bar{\Delta}/\Delta]M'/y]B'$$

and $\Gamma, \bar{\Delta}, \Delta \vdash [M'/y]B' = [[\bar{\Delta}/\Delta]M'/y]B' \div \mathsf{type}$, and also $\Gamma, \bar{\Delta}$ is disjoint from Δ, since the variables in $\bar{\Delta}$ were chosen to be fresh. By the induction hypothesis, we get

$$\Gamma, \bar{\Delta} \vdash (M_1 \cdot ([\bar{\Delta}/\Delta]M' \div \mathsf{nil})) = (M_2 \cdot ([\bar{\Delta}/\Delta]M' \div \mathsf{nil})) : [[\bar{\Delta}/\Delta]M'/y]B'$$

Using Lemma 5.1 again, we can see that

$$\Gamma, \bar{\Delta}, w \div [\bar{\Delta}/\Delta]B \vdash (M_1 \cdot (w \cdot \mathsf{nil} \div \mathsf{nil})) = (M_2 \cdot (w \cdot \mathsf{nil} \div \mathsf{nil})) : [w \cdot \mathsf{nil}/y]B'$$

By, $\mathsf{oe\_ext}$

$$\Gamma, \bar{\Delta} \vdash M_1 = M_2 : \Pi w {\div} [\bar{\Delta}/\Delta]B.[w \cdot \mathsf{nil}/y]B'$$

Lemma 5.3 provides the simplification

$$\Gamma, \bar{\Delta} \vdash M_1 = M_2 : \Pi w {\div} [\bar{\Delta}/\Delta]B.[w/y]B'$$

and α-conversion gives

$$\Gamma, \bar{\Delta} \vdash M_1 = M_2 : \Pi y {\div} [\bar{\Delta}/\Delta]B.B'$$

But we can kind convert to

$$\Gamma, \bar{\Delta}, \Delta \vdash M_1 = M_2 : \Pi y {\div} B.B'$$

and so

$$\Gamma \vdash M_1 = M_2 : A_2$$

by strengthening, using the observation that no variables in $\Delta, \bar{\Delta}$ can appear free in M_1 or M_2 by their typing.

■

5.3 Local Strict Occurrences

$$\frac{\Delta \vdash S \ \mathsf{pat} \qquad \Delta \vdash S : B > C}{\Gamma; \Delta \Vdash_x x \cdot S}\ \mathsf{ls\_pat}$$

$$\frac{\Gamma; \Delta \Vdash_x M}{\Gamma; \Delta \Vdash_x (M; S)}\ \mathsf{ls\_hd} \qquad \frac{\Gamma; \Delta \Vdash_x S}{\Gamma; \Delta \Vdash_x (M; S)}\ \mathsf{ls\_sp}$$

$$\frac{\Gamma; \Delta \Vdash_x S}{\Gamma; \Delta \Vdash_x (M \div S)}\ \mathsf{ls\_isp}$$

$$\frac{y : A \in \Delta \qquad \Gamma; \Delta \Vdash_x S}{\Gamma; \Delta \Vdash_x y \cdot S}\ \mathsf{ls\_var}$$

$$\frac{\Gamma; \Delta \Vdash_x S}{\Gamma; \Delta \Vdash_x a \cdot S}\ \mathsf{ls\_a} \qquad \frac{\Gamma; \Delta \Vdash_x S}{\Gamma; \Delta \Vdash_x c \cdot S}\ \mathsf{ls\_c}$$

$$\frac{d = M : A \in \Sigma \qquad \cdot \Vdash M \qquad \Gamma; \Delta \Vdash_x S}{\Gamma; \Delta \Vdash_x d \cdot S}\ \mathsf{ls\_d}$$

$$\frac{M \stackrel{whr}{\longrightarrow} M' \qquad \Gamma; \Delta \Vdash_x M'}{\Gamma; \Delta \Vdash_x M}\ \mathsf{ls\_red}$$

$$\frac{\Gamma; \Delta \Vdash_x A}{\Gamma; \Delta \Vdash_x \lambda y : A.M}\ \mathsf{ls\_ld} \qquad \frac{\Gamma; \Delta, y \star A \Vdash_x M}{\Gamma; \Delta \Vdash_x \lambda y \star A.M}\ \mathsf{ls\_lb}$$

$$\frac{\Gamma; \Delta \Vdash_x A_1}{\Gamma; \Delta \Vdash_x \Pi y : A_1.A_2}\ \mathsf{ls\_pd} \qquad \frac{\Gamma; \Delta, y \star A_1 \Vdash_x A_2}{\Gamma; \Delta \Vdash_x \Pi y \star A_1.A_2}\ \mathsf{ls\_pb}$$

5.4 Strict Occurrences

$$\frac{M \stackrel{whr}{\longrightarrow} M' \qquad \Gamma \Vdash_x M'}{\Gamma \Vdash_x M}\ \mathsf{rs\_red}$$

$$\frac{d = M : A \in \Sigma \qquad \cdot \Vdash M \qquad \Gamma; \cdot \Vdash_x S}{\Gamma \Vdash_x d \cdot S}\ \mathsf{rs\_d}$$

$$\frac{\Gamma; \cdot \Vdash_x S}{\Gamma \Vdash_x c \cdot S}\ \mathsf{rs\_c}$$

$$\frac{\Gamma, y \star A \Vdash_x M}{\Gamma \Vdash_x \lambda y \star A.M}\ \mathsf{rs\_lam}$$

5.5 Strictness

$$\frac{M \xrightarrow{whr} M' \qquad \Gamma \Vdash M'}{\Gamma \Vdash M}\,\mathsf{gs\_red}$$

$$\frac{d = M : A \in \Sigma \qquad \cdot \Vdash M \qquad \Gamma \Vdash M \cdot S}{\Gamma \Vdash d \cdot S}\,\mathsf{gs\_d}$$

$$\frac{}{\Gamma \Vdash c \cdot S}\,\mathsf{gs\_c}$$

$$\frac{\Gamma, x : A \Vdash_x M \qquad \Gamma, x : A \Vdash M}{\Gamma \Vdash \lambda x : A.M}\,\mathsf{gs\_lam}$$

$$\frac{\Gamma, x \div A \Vdash M}{\Gamma \Vdash \lambda x \div A.M}\,\mathsf{gs\_ilam}$$

Lemma 5.5 *(Completeness) Let σ_1, σ_2 be such that $\Gamma'; \Delta' \vdash \sigma_1, \sigma_2 : \Gamma; \Delta$.*

1. *If*
   - *(a)* $x : A \in \Gamma$
   - *(b)* $\Gamma; \Delta \Vdash_x M$
   - *(c)* $\Gamma, \Delta \vdash M : B$
   - *(d)* $\Gamma', \Delta' \vdash M[\sigma_1] = M[\sigma_2] : B[\sigma_1]$

   then $\Gamma' \vdash \sigma_1(x) = \sigma_2(x) : A[\sigma_1]$.

2. *If*
   - *(a)* $x : A \in \Gamma$
   - *(b)* $\Gamma; \Delta \Vdash_x B$
   - *(c)* $\Gamma, \Delta \vdash B : \mathsf{type}$
   - *(d)* $\Gamma', \Delta' \vdash B[\sigma_1] = B[\sigma_2] : \mathsf{type}$

   then $\Gamma' \vdash \sigma_1(x) = \sigma_2(x) : A[\sigma_1]$.

3. *If*
   - *(a)* $x : A \in \Gamma$
   - *(b)* $\Gamma; \Delta \Vdash_x S$
   - *(c)* $\Gamma, \Delta \vdash S : B > C$
   - *(d)* $\Gamma', \Delta' \vdash S[\sigma_1] = S[\sigma_2] : B[\sigma_1] > C[\sigma_1]$

   then $\Gamma' \vdash \sigma_1(x) = \sigma_2(x) : A[\sigma_1]$.

4. *If*
   - *(a)* $x : A \in \Gamma$

(b) $\Gamma \Vdash_x M$

(c) $\Gamma, \Delta \vdash M : B$

(d) $\Gamma', \Delta' \vdash S : B[\sigma_1] > C[\sigma_1]$,

(e) $\Gamma' \vdash M[\sigma_1] \cdot S = M[\sigma_2] \cdot S : C[\sigma_1]$

then $\Gamma' \vdash \sigma_1(x) = \sigma_2(x) : A[\sigma_1]$.

5. *If*

(a) $\Gamma \Vdash M$

(b) $\Gamma \vdash M : B$

(c) $\Gamma \vdash S_1, S_2 : B[\sigma_1] > C[\sigma_1]$

(d) $\Gamma' \vdash M[\sigma_1] \cdot S_1 = M[\sigma_2] \cdot S_2 : C[\sigma_1]$

then $\Gamma' \vdash S_1 = S_2 : B[\sigma_1] > C[\sigma_1]$.

Proof Simultaneous induction over the rules defining the $\Vdash$ judgments. ■

Theorem 5.6 *(Injectivity) If* $d : A = M$ *is strict, i.e.* $\cdot \Vdash M$, *then* d *is injective. (as in [PS98])*

Proof Follows immediately from Lemma 5.5 (5), with σ_1, σ_2 both the identity substitution. ■

References

[PS98] Frank Pfenning and Carsten Schürmann. Algorithms for equality and unification in the presence of notational definitions. In T. Altenkirch, W. Naraschewski, and B. Reus, editors, *Types for Proofs and Programs*, pages 179–193, Kloster Irsee, Germany, March 1998. Springer-Verlag LNCS 1657.