A Judgmental Deconstruction of Modal Logic

Jason Reed

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Abstract

The modalities $\Box$ and $\bigcirc$ of necessary and lax truth described by Pfenning and Davies can be seen to arise from the same pair of adjoint logical operators F and U, which pass in both directions between two judgments of differing strength. This may be generalized to a logic with many such adjunctions, across judgments subject to different substructral disciplines, allowing explanation of possibility $\Diamond$, linear logic's modality !, and intuitionistic labelled deduction as well.

1 Introduction

Insidious rumors may have reached your ears to the effect of

- In judgmental modal S4 [PD01] according to Pfenning and Davies, the validity judgment itself is right-asynchronous and left-synchronous
- In linear logic [Gir87], the modality ! is made of two mysterious 'half-connectives'
- The point of judgments [ML96] is to allow the same proposition to be judged in different ways

The goal of this note is to clear up the confusion: Judgments are not left- or right-asynchronous or -synchronous or anything else of the sort. $!, \Box, \bigcirc$ are each constituted from two perfectly ordinary and well-behaved logical connectives — given a certain generosity of interpretation, the *same* two.

Moreover, there is no particular need when simply defining a modal logic to have many different judgments upon exactly the same underlying logical data. Nothing *prevents* us from doing so — nothing ever prevents us from defining whatever predicates we like after the fact — but the typical judgments that encode *modes of truth* may fruitfully be arranged so that *different* modes of truth are to be predicated on entirely *different* classes of propositions. In short, it is helpful to live in a world where the sort of thing that is eligible to be true is different from the sort of thing that is eligible to be, for instance, *necessarily* true. In a slogan:

Different judgments judge different things.

But don't worry: there is still a circumstance that allows us to not lose the expressive power we thought we had a moment ago, when one and the same proposition could be proven or supposed true, necessary, possible, lax, constructible from the current set of resources, true at time t, true according to agent K, and so forth: it is the ubiquity of unary logical connectives that act as *coercions* between different judgments, i.e. different notions of truth. Indeed in everyday life we depend on some kind of transport between the propositions we utter and those uttered by our neighbors to bring them into correspondence, but, as the category theorists admonish us, we should not confuse *identity* with *isomorphism*.

And of course we should not necessarily expect every round-trip around these propositional transportations to be the identity. It becomes evident in fact that the most common and familiar modal operations are precisely the 'failure of holonomy' of certain paths among modes of truth.

2 Language

The logical language described below, call it 'adjoint logic', is parametrized by a set M of modes of truth, together with a preorder (reflexive, transitive relation) $\leq$ on M. For typical elements of M we use the letters p, q, r.

For each $p \in M$, there is a notion of proposition-at-p. Its syntax is as follows

$$A_p ::= F_{q \geq p} A_q \mid U_{q \leq p} A_q \mid A_p \wedge_p A_p \mid A_p \vee_p A_p \mid A_p \Rightarrow_p A_p \mid \top_p \mid \bot_p \mid a_p$$

The subscript p on the familiar logical connectives indicates that formally we are keeping track of *where* (i.e at which mode of truth) the conjunction, disjunction, implication is taking place. Likewise there is a separate class of atomic propositions a_p for each p. The notation $F_{q \geq p}$ and $U_{q \leq p}$ is meant to convey that *if* $q \geq p$ in the preorder structure supposed on M, then $F_{q \geq p}$ is in fact allowed to be used as a logical connective, and similarly for U with the inequality running the opposite direction.

We are careful *not* to indulge in the Martin-Löfian absurdity of saying

$$\frac{\vdash A_q \; \mathsf{prop}_q \qquad \vdash q \geq p}{\vdash F_{q \geq p} A_q \; \mathsf{prop}_p}$$

as if this defined the syntax of propostions via inference rules on the same putative footing as those that tell us how to *prove* $F_{q \geq p} A_q$, despite the absence of anything telling us where the *subject* $F_{q \geq p} A_q$ of the allegedly one-place judgment prop_p ('is a proposition-at-p') *comes* from in the first place.

Instead, if forced to used inference rules, we would much better say

$$\frac{\vdash \mathsf{prop}_q \qquad \vdash q \geq p}{\vdash \mathsf{prop}_p} F$$

reserving F for simply the *name* of the inference rule itself, and taking prop_p instead as a zero-place predicate '*there is* a proposition-at-p'. The constructive reading of ‘if there is a proposition at q, and $q \geq p$, then there is a proposition at p’ gives us precsisely what we want — the set of propositions is precisely the set of proofs that the set of propositions is inhabited.

3 Proofs

We now give a sequent calculus for adjoint logic and observe that it is internally sound and complete.

A context Γ is something of the grammar

$$\Gamma \quad ::= \quad \cdot \mid \Gamma, A_p\,\mathsf{true}_p$$

For the time being we will ignore substructural logics and suppose that all hypotheses are subject to weakening and contraction as in ordinary intuitionistic logic. Linear and other substructural logics are taken up in Section 4.4.

A sequent, the sort of thing amenable to *being provable*, is something of the form

$$\Gamma \vdash A_p\,\mathsf{true}_p$$

subject to the restriction that for every $A_q\,\mathsf{true}_q \in \Gamma$, we have $q \geq p$. Lest this requirement pass too quickly by the reader’s eyes, it should be noted that it is the central mechanism by which modalities have any force in the logic. If $\leq$ is viewed as ordering modes of truth by strength, we are positing that it *does not make sense* to think about a entailing a proposition under a certain mode of truth if it is subject to any hypotheses of a *weaker* mode of truth.

The rules of the sequent calculus are as follows, omitting the judgmental scaffolding true_p and the subscript p on connectives when the choice of p is obvious from context. They include the familiar hypothesis rule and left and right rules for all the standard connectives:

$$\frac{}{\Gamma, a_p \vdash a_p}\,hyp \qquad \frac{\Gamma \vdash A_p \qquad \Gamma \vdash B_p}{\Gamma \vdash A_p \wedge B_p}\,\wedge R \qquad \frac{\Gamma, A_p, B_p \vdash C_r}{\Gamma, A_p \wedge B_p \vdash C_r}\,\wedge L \qquad \frac{\Gamma \vdash A_p}{\Gamma \vdash A_p \vee B_p}\,\vee R_1$$

$$\frac{\Gamma \vdash B_p}{\Gamma \vdash A_p \vee B_p}\,\vee R_2 \qquad \frac{\Gamma, A_p \vdash C_r \qquad \Gamma, B_p \vdash C_r}{\Gamma, A_p \vee B_p \vdash C_r}\,\vee L \qquad \frac{\Gamma, A_p \vdash B_p}{\Gamma \vdash A_p \Rightarrow B_p} \Rightarrow R$$

$$\frac{\Gamma \vdash A_p \qquad \Gamma, B_p \vdash C_r}{\Gamma, A_p \Rightarrow B_p \vdash Cr} \Rightarrow L \qquad \frac{}{\Gamma, \bot_p \vdash C_r}\,\bot L \qquad \frac{}{\Gamma \vdash \top_p}\,\top R$$

as well as rules for F and U:

$$\frac{\Gamma \vdash A_q}{\Gamma \vdash U_{q \leq p} A_q}\,UR \qquad \frac{q \geq r \qquad \Gamma, A_q \vdash C_r}{\Gamma, U_{q \leq p} A_q \vdash C_r}\,UL \qquad \frac{\Gamma\lfloor_{\geq q} \vdash A_q}{\Gamma \vdash F_{q \geq p} A_q}\,FR \qquad \frac{\Gamma, A_q \vdash C_r}{\Gamma, F_{q \geq p} A_q \vdash C_r}\,FL$$

where $\Gamma\lfloor_{\geq q}$ means the context made of only those $A_p\,\mathsf{true}_p$ found in Γ such that $p \geq q$.

We then have

Lemma 3.1 (Cut Admissibility) *For any Γ, p, r such that $p \leq r$ and every true_q in Γ has $q \leq p$, if $\Gamma \vdash A_p$ and $\Gamma, A_p \vdash C_r$, then $\Gamma \vdash C_r$.*

Proof By induction on A_p and the relevant derivations, using standard structural cut elimination techniques [Pfe95, Pfe00] ■

and

Lemma 3.2 (Identity) *For any A_p, we have $A_p \vdash A_p$*

Proof By induction on A_p. ■

Some comments are due about the behavior of this system with respect to Andreoli-style focusing [And92]: U is a *negative* connective, left-synchronous and right-asynchronous, and F is conversely *positive*, i.e. left-asynchronous and right-synchronous. Without proving that focusing discipline is correct for the entire system, a task for another paper, we can at least observe that U is invertible on the right precisely because it moves 'with the grain' with respect to the central invariant on sequents that their right sides are $\leq$-smaller than their left, for U as it is stripped away only transports the right side of the sequent to a mode of truth that is *even smaller* by $\leq$ than it already was, which by the assumed transitivity of $\leq$ guarantees the invariant is still satisfied. We find F is invertible on the left for exactly the same reason.

4 Representations

In this section we discuss how various logics and logical features can be construed as special cases of adjoint logic.

4.1 Pfenning-Davies $\Box$

Pfenning and Davies [PD01] describe an intuitionistic alethic modal logic which, if rendered classical by the addition of suitable axioms, is equivalent to the familiar classical modal logic S4.

The entailment relation has the form

$$\Delta; \Gamma \vdash_{\mathrm{PD}} A$$

where Δ is something of the form $A_1\,\mathsf{valid}, \ldots, A_n\,\mathsf{valid}$, and Γ of the form $A_1\,\mathsf{true}, \ldots A_n\,\mathsf{true}$.

The important rules natural deduction for our purposes are introduction and elimination for $\Box$, and the use of valid hypotheses:

$$\frac{\Delta; \Gamma \vdash_{\mathrm{PD}} \Box A \qquad \Delta, A\,\mathsf{valid}; \Gamma \vdash_{\mathrm{PD}} C}{\Delta; \Gamma \vdash_{\mathrm{PD}} C} \qquad \frac{\Delta; \cdot \vdash_{\mathrm{PD}} A}{\Delta; \Gamma \vdash_{\mathrm{PD}} \Box A} \qquad \frac{}{\Delta, A\,\mathsf{valid}; \Gamma \vdash_{\mathrm{PD}} A}$$

This logic corresponds to a simple subset of adjoint logic for M being the preorder with two points, call them t and v, in which $t \leq v$ and not $v \leq t$. The subset we need contains the traditional connectives (as well as F) only at t, and the only connective at all at the mode v is U. Formally, we are only considering

$$\begin{array}{lcl} A_v & ::= & U_{t \leq v} A_t \\ A_t & ::= & F_{v \geq t} A_v \mid A_t \wedge_t A_t \mid A_t \vee_t A_t \mid A_t \Rightarrow_t A_t \mid \top_t \mid \bot_t \mid a_t \end{array}$$

Note that there is only one pertinent F and one U in this system so we can refer to them as simply F and U. Let the translation A^* of a PD proposition A be A with every $\Box$ replaced by FU and every other connective replaced by the appropriate t-subscripted analogue.

We then have (lifting operations such as $—^*$ and U to contexts in the evident way)

Theorem 4.1

- $\Delta; \cdot \vdash_{PD} A$ *iff* $U\Delta^* \vdash UA^* \; \mathsf{true}_v$
- $\Delta; \Gamma \vdash_{PD} A$ *iff* $U\Delta^*, \Gamma^* \vdash A^* \; \mathsf{true}_t$

Proof By induction on the relevant derivations, taking advantage of the fact that U is invertible on the right, the substitution principle for the natural deduction system, and identity and cut admissibility for the sequent calculus. ∎

The correspondence between $A \, \mathsf{valid}$ in the PD system and UA^* in adjoint logic reveals that the vague notion that valid was somehow 'intrinsically negative as a judgment' (and therefore compatible with left focus, and appearing only transiently on the right by dint of being asynchronous there) is really an epiphenomenon of it systematically concealing a perfectly ordinary negative connective U.

We might as well have begun by defining a 'native' sequent calculus for PD modal logic, perhaps by the rules

$$\frac{\Delta; \Gamma, A \, \mathsf{true} \vdash_{\mathrm{PD}} C}{\Delta, A \, \mathsf{valid}; \Gamma \vdash_{\mathrm{PD}} C} \qquad \frac{\Delta; \cdot \vdash_{\mathrm{PD}} A}{\Delta; \Gamma \vdash_{\mathrm{PD}} \Box A} \qquad \frac{\Delta, A \, \mathsf{valid}; \Gamma \vdash_{\mathrm{PD}} C}{\Delta; \Gamma, \Box A \vdash_{\mathrm{PD}} C}$$

in which case we can see that the process of decomposing a $\Box$ on either side of the turnstile is isomorphic to that of decomposing FU. On the left, consider

$$\frac{\Delta, A \, \mathsf{valid}; \Gamma \vdash C}{\Delta; \Gamma, \Box A \, \mathsf{true} \vdash C} \quad \Longleftrightarrow \quad \frac{\Gamma, UA^* \, \mathsf{true}_v \vdash C^*}{\Gamma, FUA^* \, \mathsf{true}_t \vdash C^*}$$

and furthermore the erstwhile structural 'copy' rule becomes simply the left rule for U.

$$\frac{\Delta; \Gamma, A \, \mathsf{true} \vdash C}{\Delta, A \, \mathsf{valid}; \Gamma \vdash C} \quad \Longleftrightarrow \quad \frac{\Gamma, A^* \, \mathsf{true}_t \vdash C^*}{\Gamma, UA^* \, \mathsf{true}_v \vdash C^*}$$

Meanwhile on the right we see the correspondence

$$\frac{\Delta; \cdot \vdash A\,\mathsf{true}}{\Delta; \Gamma \vdash \Box A\,\mathsf{true}} \qquad \Longleftrightarrow \qquad \frac{\dfrac{\Gamma\lfloor_{\geq v} \vdash C^*\,\mathsf{true}_t}{\Gamma\lfloor_{\geq v} \vdash UC^*\,\mathsf{true}_v}}{\Gamma \vdash FUC^*\,\mathsf{true}_t}$$

where the forced sequencing on the right is justified by essentially focusing reasoning since U is right-asynchronous — once we decompose the F there is no reason not to continue decomposing the U.

4.2 Pfenning-Davies $\bigcirc$

The Pfenning-Davies account of lax logic (also found in [PD01]) is concerned with a different modality $\bigcirc$, defined by allowing the entailment to be one of the two forms

$$\Gamma \vdash_{\mathrm{PD}} A\,\mathsf{true} \qquad \Gamma \vdash_{\mathrm{PD}} A\,\mathsf{lax}$$

for Γ consisting only of hypotheses of the form $A\,\mathsf{true}$, and giving the rules

$$\frac{\Gamma \vdash \bigcirc A \qquad \Gamma, A \vdash_{\mathrm{PD}} C\,\mathsf{lax}}{\Gamma \vdash_{\mathrm{PD}} C\,\mathsf{lax}} \qquad \frac{\Gamma \vdash_{\mathrm{PD}} A\,\mathsf{lax}}{\Gamma \vdash_{\mathrm{PD}} \bigcirc A\,\mathsf{true}} \qquad \frac{\Gamma \vdash_{\mathrm{PD}} A\,\mathsf{true}}{\Gamma \vdash_{\mathrm{PD}} A\,\mathsf{lax}}$$

Somewhat remarkably, the subset required for encoding $\bigcirc$ is the same as that for $\Box$ but upside-down. We again take the two-point preorder, this time calling the two points $\ell \leq t$ (though it should be noted that the names do not actually matter!) and inhabiting only the mode t with most of the connectives:

$$\begin{array}{lcl} A_t & ::= & U_{\ell \leq t} A_\ell \mid A_t \wedge_t A_t \mid A_t \vee_t A_t \mid A_t \Rightarrow_t A_t \mid \top_t \mid \bot_t \mid a_t \\ A_\ell & ::= & F_{t \geq \ell} A_t \end{array}$$

The translation in this case requires that A^* replaces every occurrence of $\bigcirc$ with UF, and every other connective with its t-subscripted twin. The theorem that realizes the encoding's adequacy is

Theorem 4.2

- $\Gamma \vdash_{PD} A\,\mathsf{true}$ *iff* $\Gamma^* \vdash A^*$
- $\Gamma \vdash_{PD} A\,\mathsf{lax}$ *iff* $\Gamma^* \vdash FA^*$
- $\Gamma, A \vdash_{PD} C\,\mathsf{lax}$ *iff* $\Gamma^*, FA^* \vdash FC^*$

Proof By induction on the relevant derivations, taking advantage of the fact that F is invertible on the left, the substitution principle for the natural deduction system, and identity and cut admissibility for the sequent calculus. ■

Here we find that the 'structural' rule that allows us to infer $A\,\mathsf{true}$ from m $A\,\mathsf{lax}$ is none other than the right rule for the connective F.

It is perspicuous again to consider the 'native' sequent calculus rules for the PD lax modality, namely

$$\frac{\Gamma, A \vdash_{\mathrm{PD}} C\ \mathsf{lax}}{\Gamma, \bigcirc A \vdash_{\mathrm{PD}} C\ \mathsf{lax}} \qquad \frac{\Gamma \vdash_{\mathrm{PD}} A\ \mathsf{lax}}{\Gamma \vdash_{\mathrm{PD}} \bigcirc A\ \mathsf{true}} \qquad \frac{\Gamma \vdash_{\mathrm{PD}} A\ \mathsf{true}}{\Gamma \vdash_{\mathrm{PD}} A\ \mathsf{lax}}$$

and identify the relationships

$$\frac{\Gamma, A \vdash_{\mathrm{PD}} C\ \mathsf{lax}}{\Gamma, \bigcirc A \vdash_{\mathrm{PD}} C\ \mathsf{lax}} \iff \cfrac{\cfrac{\Gamma^*, A^* \vdash FC^*}{\Gamma^*, FA^* \vdash FC^*}}{\Gamma^*, UFA^* \vdash FC^*}$$

$$\frac{\Gamma \vdash_{\mathrm{PD}} A\ \mathsf{lax}}{\Gamma \vdash_{\mathrm{PD}} \bigcirc A\ \mathsf{true}} \iff \frac{\Gamma^* \vdash FA^*}{\Gamma^* \vdash UFA^*}$$

$$\frac{\Gamma \vdash_{\mathrm{PD}} A\ \mathsf{true}}{\Gamma \vdash_{\mathrm{PD}} A\ \mathsf{lax}} \iff \frac{\Gamma^* \vdash A^*}{\Gamma^* \vdash FA^*}$$

between partial derivations in PD and its encoding.

4.3 Multimodal Logics

A multimodal logic with many $\Box$s of differing strengths corresponds simply to having a large preorder for M, and assigning basic 'ordinary truth' to its least element (at which all ordinary connectives are defined), and each $\Box$ a round-trip of the form FU up to some high strength mode of truth, and back down to 'ordinary truth'.

However adjoint logic does not *require* this stereotypical setup where there is a single distinguished mode of truth that is 'basic' but rather allows all connectives to be defined at every mode, and implicitly allows a different $\Box$ and different $\bigcirc$ for every 'round trip through a higher mode' and 'round trip through a lower mode' respectively.

4.4 Linear Logic with !

We may extend adjoint logic with substructural features by allowing a specification for each mode of truth of which structural rules it is required to satisfy, so long as if $p \leq q$, we have that any structural rule satisfied by p is also satisfied by q. This is so that, for instance, $F_{q \geq p}$ remains correctly left-invertible. Otherwise, it might be that one would like to apply structural rules at mode p before (in a bottom-up reading) moving via F to mode q where those structural rules are no longer available, meaning that proof search incorporating eager decomposition of F would not be complete.

To accomodate substructrual properties we must slightly generalize the right rule for F to be the following

$$\frac{\Gamma \rightsquigarrow \Gamma_{\geq q} \qquad \Gamma_{\geq q} \vdash A_q}{\Gamma \vdash F_{q \geq p} A_q}\, FR$$

where $\Gamma \rightsquigarrow \Gamma_{\geq q}$ means that Γ can be converted to $\Gamma_{\geq q}$ via allowed structural rules, and in fact $\Gamma_{\geq q}$ is a context containing only judgments true_p where $p \geq q$. In this way we allow, for instance, weakening of hypotheses at modes that were marked as allowing weakening, but we cannot apply the F right rule at all until all unweakenable hypotheses have been eliminated.

Having done this we can now encode linear logic with !, which winds up unsurprisingly being very similar to PD $\Box$. The subset of adjoint logic required is again a two-point M with $r \leq u$, (for **r**esources and **u**nrestricted hypotheses) where at now we say that at u we allow weakening and contraction, and at r we do not. The connectives used are

$$
\begin{array}{lcl}
A_u & ::= & U_{r\leq u}A_r \\
A_r & ::= & F_{u\geq r}A_u \mid A_r \&_r A_r \mid A_r \oplus_r A_r \mid A_r \otimes_r A_r \mid A_r \multimap_r A_r \mid \\
 & & \top_r \mid 0_r \mid 1_r \mid a_r
\end{array}
$$

where the linear connectives in adjoint logic have the evident rules identical to those from linear logic except generalized to adjoint logic contexts and conclusions.

To embed linear logic with entailments $\Delta; \Gamma \vdash_{\mathrm{LL}} A$ where Δ is full of unrestricted assumptions and Γ linear resources, we say that A^* replaces every ! in A with FU and again subscripts every other connective appropriately, and check

Theorem 4.3

- $\Delta; \cdot \vdash_{LL} A$ *iff* $U\Delta^* \vdash UA^* \ \mathsf{true}_u$
- $\Delta; \Gamma \vdash_{LL} A$ *iff* $U\Delta^*, \Gamma^* \vdash A^* \ \mathsf{true}_r$

One distinct advantage of treating linear logic in this way is that we are able to smoothly incorporate the connectives of nonlinear intuitionistic logic in the same system. They may simply be added as connectives native to the mode of truth u, leaving us with the following adjoint logic

$$
\begin{array}{lcl}
A_u & ::= & U_{r\leq u}A_r \mid A_u \wedge_u A_u \mid A_u \vee_u A_u \mid A_u \Rightarrow_u A_u \mid \top_u \mid \bot_u \mid a_u \\
A_r & ::= & F_{u\geq r}A_u \mid A_r \&_r A_r \mid A_r \oplus_r A_r \mid A_r \otimes_r A_r \mid A_r \multimap_r A_r \mid \\
 & & \top_r \mid 0_r \mid 1_r \mid a_r
\end{array}
$$

In it we can conveniently see directly by construction of small prooftrees that, for instance, F commutes with positive connectives and U with negative connectives:

$$
\begin{array}{cc}
FA \otimes FB \dashv\vdash F(A \wedge B) & UA \wedge UB \dashv\vdash U(A\&B) \\
FA \oplus FB \dashv\vdash F(A \vee B) & A \Rightarrow UB \dashv\vdash U(FA \multimap B) \\
1 \dashv\vdash F\top & \top_u \dashv\vdash U\top_r \\
0 \dashv\vdash F\bot &
\end{array}
$$

We can then derive more familiar identities involving ! such as $!A \otimes !B \dashv\vdash !(A\&B)$ because $FUA \otimes FUA \dashv\vdash F(UA \wedge UB) \dashv\vdash FU(A\&B)$. Seeing how ! separated into positive F and negative U, we can see this arises directly from the ambipolarity of $\wedge$ in nonlinear intuitionistic logic. In the same way, we are also able to see more clearly why $\bigcirc(A \wedge B) \dashv\vdash \bigcirc A \wedge \bigcirc B$ in lax logic, but not, for instance, $\bigcirc(A \vee B) \dashv\vdash \bigcirc A \vee \bigcirc B$ or $\bigcirc A \Rightarrow \bigcirc B \vdash \bigcirc(\bigcirc A \Rightarrow B)$, even though $F(A \vee_t B) \dashv\vdash FA \vee_\ell FB$ and $U(FA \Rightarrow_\ell B) \dashv\vdash (A \Rightarrow_t UB)$ if we bother to include 'natively lax' connectives $\Rightarrow_\ell$ and $\vee_\ell$.

4.5 Pfenning-Davies $\diamond$

Deepak Garg noted (personal communication) that lax logic can also be encoded in linear logic via the definition $\bigcirc A = (A \multimap a) \multimap a$ for a a fresh atom. We can represent $\diamond$ similarly as a 'parameteric De Morgan dual of $\Box$' (see also [CCP03] for other examples of parametric translations in linear logic) interposing a PD $\Box$ between the two 'negations' — $\multimap a$ and making the definition $\diamond A = (\Box(A \multimap a)) \multimap a$. Subsequently we may reuse our interpretation above of $\Box$ as FU.

To achieve this, however, we need a notion of hypotheses that are at once linear, to maintain the intuitionistic character of the logic[1], and somehow 'more valid' than ordinary linear hypotheses, to achieve the context-clearing effect of the PD elimination rule for $\diamond$. We cannot use the notion of validity already in the logic, since it is not linear, but fortunately the generality of the adjoint logic easily permits introducing a mode of truth 'more valid than' another, and requiring that it behaves linearly.

First let us recall the PD natural deduction calculus for $\diamond$. There are valid contexts Δ and true contexts Γ, and two entailments,

$$\Delta; \Gamma \vdash_{\mathrm{PD}} A\,\mathsf{true} \qquad \Delta; \Gamma \vdash_{\mathrm{PD}} A\,\mathsf{poss}$$

and rules governing poss and $\diamond$

$$\frac{\Delta; \Gamma \vdash \diamond A \qquad \Delta; A \vdash_{\mathrm{PD}} C\,\mathsf{poss}}{\Delta; \Gamma \vdash_{\mathrm{PD}} C\,\mathsf{poss}} \qquad \frac{\Delta; \Gamma \vdash_{\mathrm{PD}} A\,\mathsf{poss}}{\Delta; \Gamma \vdash_{\mathrm{PD}} \diamond A\,\mathsf{true}} \qquad \frac{\Delta; \Gamma \vdash_{\mathrm{PD}} A\,\mathsf{true}}{\Delta; \Gamma \vdash_{\mathrm{PD}} A\,\mathsf{poss}}$$

Note that Γ is erased in the second premise of the elimination rule. In sequent form this erasure appears in the left rule as

$$\frac{\Delta; A \vdash_{\mathrm{PD}} C\,\mathsf{poss}}{\Delta; \Gamma, \diamond A \vdash_{\mathrm{PD}} C\,\mathsf{poss}}$$

To encode this logic we take adjoint logic with M a four-point[2] preorder $\{r, s, u, v\}$, with $r \leq \{u, s\} \leq v$, and allow contraction and weakening only at v and u. The connectives we need are

$$\begin{array}{lcl} A_v & ::= & U_{u \leq v} A_u \\ A_u & ::= & F_{v \geq u} A_v \mid U_{r \leq u} A_r \mid A_u \wedge_u A_u \mid \cdots \\ A_s & ::= & U_{r \leq s} A_r \\ A_r & ::= & F_{u \geq r} A_u \mid F_{s \geq r} A_s \mid A_r \multimap A_r \mid a_r \end{array}$$

[1] Otherwise 'possibility continuations' in the context would overstay their welcome.

[2] And in fact diamond-shaped!

and subsequently the definition of the modalities are given by giving clauses for translation

$$\begin{array}{rl} (\Diamond A)^* = & U_{r \le u}((F_{s \ge r} U_{r \le s}(F_{u \ge r} A^* \multimap a_r)) \multimap a_r) \\ (\Box A)^* = & F_{v \ge u} U_{u \le v} A^* \end{array}$$

We can see that $\Diamond$ still consists of only two focusing monopoles, one that begins negative on the outside, switches to positive without interruption through the outer $\multimap$, which is interrupted between $F_{s \ge r}$ and $U_{r \le s}$, and then begins another negative stretch which continues through the other $\multimap$ and switches smoothly to positive. In other words, we could have said

$$\begin{array}{rl} (\Diamond_1 A)^* = & U_{r \le u}(F_{s \ge r} A^* \multimap a_r) \\ (\Diamond_2 A)^* = & U_{r \le s}(F_{u \ge r} A^* \multimap a_r) \end{array}$$

and then $(\Diamond A)^* = (\Diamond_1 \Diamond_2 A)^*$.

The correspondence between sequent derivations before and after translation obeys

$$\begin{array}{rcl} \Delta; \Gamma \vdash A\, \mathsf{true} & \iff & U_{u \le v} \Delta^*, \Gamma^* \vdash A^*\, \mathsf{true}_u \\ \Delta; \Gamma \vdash A\, \mathsf{poss} & \iff & U_{u \le v} \Delta^*, \Gamma^*, (\Diamond_2 A)^*\, \mathsf{true}_s \vdash a_r\, \mathsf{true}_r \end{array}$$

and we can see the correspondence of partial derivations

$$\dfrac{\Delta; A \vdash_{\mathrm{PD}} C\, \mathsf{poss}}{\Delta; \Gamma, \Diamond A \vdash_{\mathrm{PD}} C\, \mathsf{poss}} \iff \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{U_{u \le v} \Delta^*, A^*\, \mathsf{true}_u, (\Diamond_2 C)^* \vdash a_r}{U_{u \le v} \Delta^*, (\Diamond_2 C)^*, F_{u \ge r} A^*\, \mathsf{true}_r \vdash a_r}}{U_{u \le v} \Delta^*, (\Diamond_2 C)^* \vdash F_{u \ge r} A^* \multimap a_r\, \mathsf{true}_r}}{U_{u \le v} \Delta^*, (\Diamond_2 C)^* \vdash (\Diamond_2 A)^*\, \mathsf{true}_s}}{\cdots \vdash F_{s \ge r}(\Diamond_2 A)^*\, \mathsf{true}_r} \quad \dfrac{}{a_r \vdash a_r}}{\dfrac{\cdots, F_{s \ge r}(\Diamond_2 A)^* \multimap a_r\, \mathsf{true}_r \vdash a_r}{U_{u \le v} \Delta^*, \Gamma^*, (\Diamond A)^*, (\Diamond_2 C)^* \vdash a_r}}$$

$$\dfrac{\Delta; \Gamma \vdash_{\mathrm{PD}} A\, \mathsf{poss}}{\Delta; \Gamma \vdash_{\mathrm{PD}} \Diamond A\, \mathsf{true}} \iff \dfrac{\dfrac{\dfrac{U_{u \le v} \Delta^*, \Gamma^*, (\Diamond_2 A)^*\, \mathsf{true}_s \vdash a_r\, \mathsf{true}_r}{U_{u \le v} \Delta^*, \Gamma^*, F_{s \ge r}(\Diamond_2 A)^*\, \mathsf{true}_r \vdash a_r\, \mathsf{true}_r}}{U_{u \le v} \Delta^*, \Gamma^* \vdash F_{s \ge r}(\Diamond_2 A)^* \multimap a_r\, \mathsf{true}_r}}{U_{u \le v} \Delta^*, \Gamma^* \vdash (\Diamond_1 \Diamond_2 A)^*\, \mathsf{true}_u}$$

$$\dfrac{\Delta; \Gamma \vdash_{\mathrm{PD}} A\, \mathsf{true}}{\Delta; \Gamma \vdash_{\mathrm{PD}} A\, \mathsf{poss}} \iff \dfrac{\dfrac{\dfrac{\dfrac{}{U_{u \le v} \Delta^*, \Gamma^* \vdash A^*\, \mathsf{true}_u}}{U_{u \le v} \Delta^*, \Gamma^* \vdash F_{u \ge r} A^*\, \mathsf{true}_r} \quad \dfrac{}{a_r \vdash a_r}}{U_{u \le v} \Delta^*, \Gamma^*, F_{u \ge r} A^* \multimap a_r\, \mathsf{true}_r \vdash a_r}}{U_{u \le v} \Delta^*, \Gamma^*, (\Diamond_2 A)^*\, \mathsf{true}_s \vdash a_r\, \mathsf{true}_r}$$

Requiring the sequencing of translated derivations to take place as depicted requires focusing reasoning beyond the scope of this note. It's possible the reasoning could be simplified by directly defining $\Diamond_1$ and $\Diamond_2$ as appropriate coalesced connectives in the adjoint logic.

4.6 Intuitionistic Labelled Deduction

Finally, consider a labelled deduction sequent calculus system with an entailment relation $\Gamma \vdash A[p]$ where Γ consists of a set hypotheses also of the form $A[p]$. The worlds p are drawn from a set M.

All ordinary logical connectives such as $\wedge$ exist and have rules that pass along the 'world part' $[p]$ of the entailment unmolested, i.e.

$$\frac{\Gamma \vdash A[p] \qquad \Gamma \vdash B[p]}{\Gamma \vdash A \wedge B[p]} \qquad \frac{\Gamma, A[p], B[p] \vdash C[r]}{\Gamma, A[p] \vdash C[r]}$$

and it possesses a connective $@_p$ with rules

$$\frac{\Gamma \vdash A[q]}{\Gamma \vdash @_q A[p]} \qquad \frac{\Gamma, A[q] \vdash C[r]}{\Gamma, @_q A[p] \vdash C[r]}$$

Then this is just the special case of adjoint logic where the relation on M is entire, i.e. $p \leq q$ for every p, q. The connective $@_p$ is equivalently translated (when 'at world q') as $F_{p \geq q}$ or $U_{p \leq q}$. Since no pair of modes of truth fail to be connected, no modal restriction obtains, and $@_p$ is ambipolar.

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