Eliminating `letcc` from Labelled Natural Deduction Proofs

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1 Language

Types:

$$A, B ::= P \mid A \wedge B \mid A \vee B \mid A \supset B \mid \top \mid \bot$$

Terms:

$$M, N ::= \langle M_1, M_2 \rangle \mid \pi_i M \mid \mathbf{inj}_i\, M$$
$$\mid (\mathbf{case}_p\, M\, \mathbf{of}\, x_1.M_1 \mid x_2.M_2)$$
$$\mid \lambda_a x.M \mid M_1\ M_2 \mid \langle\rangle \mid \mathbf{abort}_q\, M$$
$$\mid \mathbf{letcc}\, u\, \mathbf{in}\, M \mid \mathbf{throw}\, M\, \mathbf{to}\, u$$

Answers:

$$\alpha ::= M{\downarrow} \mid M \triangleright u \mid (M_p ? x_1.\alpha_1 \mid x_2.\alpha_2) \mid M!_p$$

Pre-answers:

$$\beta ::= \langle \alpha_1, \alpha_2 \rangle \mid \pi_i \alpha \mid \mathbf{inj}_i\, \alpha$$
$$\mid (\mathbf{case}_p\, \alpha\, \mathbf{of}\, x_1.\alpha_1 \mid x_2.\alpha_2)$$
$$\mid \lambda_a x.\alpha \mid \alpha_1\ \alpha_2 \mid \langle\rangle \mid \mathbf{abort}_q\, \alpha$$
$$\mid \mathbf{letcc}\, u\, \mathbf{in}\, \alpha \mid \mathbf{throw}\, \alpha\, \mathbf{to}\, u$$

1.1 Refinments of the Syntactic Class of Answers

a-Pure Answers:

$$\alpha^a ::= M{\downarrow} \mid (M?_a x_1.\alpha_1^a \mid x_2.\alpha_2^a) \mid M!_a$$

a-Separated Answers: (for $q \not\geq a$)

$$\alpha^{\bar{a}} ::= \alpha^a \mid M \triangleright u \mid (M?_q x_1.\alpha_1^{\bar{a}} \mid x_2.\alpha_2^{\bar{a}}) \mid M!_q$$

a-Pre-separated Answers:

$$\beta^a ::= (M?_a x_1.\alpha_1^{\bar{a}} \mid x_2.\alpha_2^{\bar{a}})$$

2 Typing

2.1 Terms and Pre-answers

$$X ::= M \mid \alpha$$

$$\frac{x : A[p] \in \Gamma \qquad \Gamma \vdash p \leq q}{\Gamma; \Delta \vdash x : A[q]}$$

$$\frac{\Gamma; \Delta \vdash X_1 : A[p] \qquad \Gamma; \Delta \vdash X_2 : B[p]}{\Gamma; \Delta \vdash \langle X_1, X_2 \rangle : A \wedge B[p]}$$

$$\frac{\Gamma; \Delta \vdash X : A_1 \wedge A_2[p]}{\Gamma; \Delta \vdash \pi_i X : A_i[p]}$$

$$\frac{\Gamma; \Delta \vdash X : A_i[p]}{\Gamma; \Delta \vdash \mathbf{inj}_i\, X : A_1 \vee A_2[p]}$$

$$\frac{\Gamma; \Delta \vdash X : C_1 \vee C_2[q] \qquad \begin{array}{c}\Gamma, x_1 : C_1[q]; \Delta \vdash X_1 : A[p] \\ \Gamma, x_2 : C_2[q]; \Delta \vdash X_2 : A[p]\end{array}}{\Gamma; \Delta \vdash (\mathbf{case}_q\, X\, \mathbf{of}\, x_1.X_1 \mid X_2.\alpha_2) : A[p]}$$

$$\frac{\Gamma, x : A[a], a \geq p; \Delta \vdash M : B[a]}{\Gamma; \Delta \vdash \lambda_a x.M : A \to B[p]}$$

$$\frac{\Gamma, x : A[a], a \geq p; \Delta \vdash \alpha^{\bar{a}} : B[a]}{\Gamma; \Delta \vdash \lambda_a x.\alpha^{\bar{a}} : A \to B[p]}$$

$$\frac{\Gamma; \Delta \vdash X_1 : A \supset B[p] \qquad \Gamma; \Delta \vdash X_2 : A[q] \qquad \Gamma \vdash q \geq p}{\Gamma; \Delta \vdash X_1\ X_2 : B[q]}$$

$$\frac{}{\Gamma; \Delta \vdash \langle\rangle : \top[p]}$$

$$\frac{\Gamma; \Delta \vdash X : \bot[q]}{\Gamma; \Delta \vdash \mathbf{abort}_q\, X : A[p]}$$

2.2 Answers

$$\frac{\Gamma\lfloor_{\leq p} \vdash M : A}{\Gamma; \Delta \vdash M{\downarrow} : A[p]}$$

$$\frac{\Gamma\lfloor_{\leq q} \vdash M : \bot}{\Gamma; \Delta \vdash M!_q : A[p]}$$

$$\frac{\Gamma\lfloor_{\leq q} \vdash M : B \qquad u : B[q] \in \Delta}{\Gamma; \Delta \vdash M \triangleright u : A[p]}$$

$$\frac{\Gamma\lfloor_{\leq q} \vdash M : C_1 \vee C_2 \qquad \begin{array}{c}\Gamma, x_1 : C_1[q]; \Delta \vdash \alpha_1 : A[p] \\ \Gamma, x_2 : C_2[q]; \Delta \vdash \alpha_2 : A[p]\end{array}}{\Gamma; \Delta \vdash (M?_q x_1.\alpha_1 \mid x_2.\alpha_2) : A[p]}$$

3 Termification

$$\boxed{\alpha^a \dashrightarrow_T M}$$

When J_1, J_2 are judgments and R is a relation, the symbol

$$\mathbf{PP}\begin{cases} J_1(X) \\ X \ R \ Y \\ J_2(Y) \end{cases}$$

means that both of the following are true:

- ('progress') If $J_1(X)$, then there is a Y such that $X \ R \ Y$.
- ('preservation') If $J_1(X)$ and $X \ R \ Y$, then $J_2(Y)$.

For instance, we claim of the termification relation that it satisfies

$$\mathbf{PP}\begin{cases} \Gamma, \vec{x} : \vec{A}[a], a \geq p; \Delta \vdash \alpha^a : B[a] \\ \alpha_a \dashrightarrow_T M \\ \Gamma\lfloor_{\leq p}, \vec{x} : \vec{A} \vdash M : B \end{cases}$$

(assuming that $a \notin \Gamma, \Delta$) when defined by

$$\frac{}{M\downarrow \dashrightarrow_T M}$$

$$\frac{}{M!_a \dashrightarrow_T \mathbf{abort}\, M}$$

$$\frac{\alpha_1^a \dashrightarrow_T M_1 \qquad \alpha_2^a \dashrightarrow_T M_2}{M?_a x_1.\alpha_1^a \mid x_2.\alpha_2^a \dashrightarrow_T \mathbf{case}\, M \,\mathbf{of}\, M_1 \mid M_2}$$

4 Separation

$$\boxed{\beta^a \dashrightarrow_S \alpha^{\bar{a}}}$$

$$\mathbf{PP}\begin{cases}\Gamma;\Delta \vdash \beta^a : A[p]\\ \beta^a \dashrightarrow_S \alpha^{\bar{a}}\\ \Gamma;\Delta \vdash \alpha^{\bar{a}} : A[p]\end{cases}$$

$$\frac{M?_a x_1.\alpha_{11}^{\bar{a}} \mid x_2.\alpha_2^{\bar{a}} \dashrightarrow_S M_1' \qquad M?_a x_1.\alpha_{12}^{\bar{a}} \mid x_2.\alpha_2^{\bar{a}} \dashrightarrow_S M_2'}{(M?_a x_1.(N?_q x_{11}.\alpha_{11}^{\bar{a}} \mid x_{12}.\alpha_{12}^{\bar{a}}) \mid x_2.\alpha_2^{\bar{a}}) \dashrightarrow_S N?_q x_{11}.M_1' \mid x_{12}.M_2'}$$

$$\frac{M?_a x_1.\alpha_1^{\bar{a}} \mid x_2.\alpha_{21}^{\bar{a}} \dashrightarrow_S M_1' \qquad M?_a x_1.\alpha_1^{\bar{a}} \mid x_2.\alpha_{22}^{\bar{a}} \dashrightarrow_S M_2'}{(M?_a x_1.\alpha_1^{\bar{a}} \mid x_2.(N?_q x_{21}.\alpha_{21}^{\bar{a}} \mid x_{22}.\alpha_{22}^{\bar{a}})) \dashrightarrow_S N?_q x_{21}.M_1' \mid x_{22}.M_2'}$$

$$\frac{}{(M?_a x_1.N!_q \mid x_2.\alpha_2^{\bar{a}}) \dashrightarrow_S N!_q} \qquad \frac{}{(M?_a x_1.\alpha_1^{\bar{a}} \mid x_2.N!_q) \dashrightarrow_S N!_q}$$

$$\frac{}{(M?_a x_1.N \triangleright u \mid x_2.\alpha_2^{\bar{a}}) \dashrightarrow_S N \triangleright u} \qquad \frac{}{(M?_a x_1.\alpha_1^{\bar{a}} \mid x_2.N \triangleright u) \dashrightarrow_S N \triangleright u}$$

$$\frac{}{(M?_a x_1.\alpha_1^a \mid x_2.\alpha_2^a) \dashrightarrow_S (M?_a x_1.\alpha_1^a \mid x_2.\alpha_2^a)}$$

$$\boxed{\alpha \hookrightarrow_S \alpha^{\bar{a}}}$$

$$\mathbf{PP}\begin{cases}\Gamma;\Delta \vdash \alpha' : A[p]\\ \alpha' \dashrightarrow_S \alpha^{\bar{a}}\\ \Gamma;\Delta \vdash \alpha^{\bar{a}} : A[p]\end{cases}$$

$$\frac{}{M{\downarrow} \hookrightarrow_S M{\downarrow}} \qquad \frac{}{M!_q \hookrightarrow_S M!_q} \qquad \frac{}{M \triangleright u \hookrightarrow_S M \triangleright u}$$

$$\frac{\alpha_1' \hookrightarrow_S \alpha_1^{\bar{a}} \qquad \alpha_2' \hookrightarrow_S \alpha_2^{\bar{a}} \qquad q \not\geq a}{M?_q x_1.\alpha_1' \mid x_2.\alpha_2' \hookrightarrow_S M?_q x_1.\alpha_1^{\bar{a}} \mid x_2.\alpha_2^{\bar{a}}}$$

$$\frac{\alpha_1' \hookrightarrow_S \alpha_1^{\bar{a}} \qquad \alpha_2' \hookrightarrow_S \alpha_2^{\bar{a}} \qquad M?_a x_1.\alpha_1^{\bar{a}} \mid x_2.\alpha_2^{\bar{a}} \dashrightarrow_S \alpha^{\bar{a}}}{M?_a x_1.\alpha_1' \mid x_2.\alpha_2' \hookrightarrow_S \alpha^{\bar{a}}}$$

5 `letcc` Elimination

$$\boxed{\beta \dashrightarrow \alpha}$$

$$\mathbf{PP}\begin{cases}\Gamma;\Delta \vdash \beta : A[p] \\ \beta \dashrightarrow \alpha \\ \Gamma;\Delta \vdash \alpha : A[p]\end{cases}$$

$$\boxed{M \hookrightarrow \alpha}$$

$$\mathbf{PP}\begin{cases}\Gamma;\Delta \vdash M : A[p] \\ M \hookrightarrow \alpha \\ \Gamma;\Delta \vdash \alpha : A[p]\end{cases}$$

5.1 Conjunction Introduction

$$\frac{M_1 \hookrightarrow \alpha_1 \qquad M_2 \hookrightarrow \alpha_2 \qquad \langle \alpha_1, \alpha_2 \rangle \dashrightarrow \alpha'}{\langle M_1, M_2 \rangle \hookrightarrow \alpha'}$$

$$\frac{}{\langle M_1\!\downarrow, M_2\!\downarrow \rangle \dashrightarrow \langle M_1, M_2 \rangle\!\downarrow}$$

$$\frac{}{\langle M!_q, \alpha \rangle \dashrightarrow M!_q} \qquad \frac{}{\langle \alpha, M!_q \rangle \dashrightarrow M!_q}$$

$$\frac{}{\langle M \triangleright u, \alpha \rangle \dashrightarrow M \triangleright u} \qquad \frac{}{\langle \alpha, M \triangleright u \rangle \dashrightarrow M \triangleright u}$$

$$\frac{\langle \alpha_1, \alpha \rangle \dashrightarrow \alpha_1' \qquad \langle \alpha_2, \alpha \rangle \dashrightarrow \alpha_2'}{\langle (M?_q x_1.\alpha_1 \mid x_2.\alpha_2), \alpha \rangle \dashrightarrow M?_q x_1.\alpha_1' \mid x_2.\alpha_2'}$$

$$\frac{\langle \alpha, \alpha_1 \rangle \dashrightarrow \alpha_1' \qquad \langle \alpha, \alpha_2 \rangle \dashrightarrow \alpha_2'}{\langle \alpha, (M?_q x_1.\alpha_1 \mid x_2.\alpha_2) \rangle \dashrightarrow M?_q x_1.\alpha_1' \mid x_2.\alpha_2'}$$

5.2 Conjunction Elimination

$$\frac{M \hookrightarrow \alpha \qquad \pi_i \alpha \dashrightarrow \alpha'}{\pi_i M \hookrightarrow \alpha'}$$

$$\frac{}{\pi_i(M\!\downarrow) \dashrightarrow (\pi_i M)\!\downarrow} \qquad \frac{}{\pi_i(M!_q) \dashrightarrow M!_q} \qquad \frac{}{\pi_i(M \triangleright u) \dashrightarrow M \triangleright u}$$

$$\frac{\pi_i \alpha_1 \dashrightarrow \alpha_1' \qquad \pi_i \alpha_2 \dashrightarrow \alpha_2'}{\pi_i(M?_q x_1.\alpha_1 \mid x_2.\alpha_2) \dashrightarrow M?_q x_1.\alpha_1' \mid x_2.\alpha_2'}$$

5.3 Disjunction Introduction

$$\frac{M \hookrightarrow \alpha \qquad \mathbf{inj}_i\, \alpha \dashrightarrow \alpha'}{\mathbf{inj}_i\, M \hookrightarrow \alpha'}$$

$$\frac{}{\mathbf{inj}_i(M{\downarrow}) \dashrightarrow (\mathbf{inj}_i\, M){\downarrow}} \qquad \frac{}{\mathbf{inj}_i(M!_q) \dashrightarrow M!_q} \qquad \frac{}{\mathbf{inj}_i(M \triangleright u) \dashrightarrow M \triangleright u}$$

$$\frac{\mathbf{inj}_i\, \alpha_1 \dashrightarrow \alpha_1' \qquad \mathbf{inj}_i\, \alpha_2 \dashrightarrow \alpha_2'}{\mathbf{inj}_i(M?_q x_1.\alpha_1 \mid x_2.\alpha_2) \dashrightarrow M?_q x_1.\alpha_1' \mid x_2.\alpha_2'}$$

5.4 Disjunction Elimination

$$\frac{M \hookrightarrow \alpha \qquad M_1 \hookrightarrow \alpha_1 \qquad M_2 \hookrightarrow \alpha_2 \qquad \mathbf{case}_q\, \alpha\, \mathbf{of}\, x_1.\alpha_1 \mid x_2.\alpha_2 \dashrightarrow \alpha'}{\mathbf{case}_q\, M\, \mathbf{of}\, x_1.M_1 \mid x_2.M_2 \hookrightarrow \alpha'}$$

$$\frac{}{\mathbf{case}_q\, M!\, \mathbf{of}\, x_1.\alpha_1 \mid x_2.\alpha_2 \dashrightarrow M!}$$

$$\frac{}{\mathbf{case}_q\, M \triangleright u\, \mathbf{of}\, x_1.\alpha_1 \mid x_2.\alpha_2 \dashrightarrow M \triangleright u}$$

$$\frac{\mathbf{case}_q\, \alpha_1'\, \mathbf{of}\, x_1.\alpha_1 \mid x_2.\alpha_2 \dashrightarrow \alpha_1'' \qquad \mathbf{case}_q\, \alpha_2'\, \mathbf{of}\, x_1.\alpha_1 \mid x_2.\alpha_2 \dashrightarrow \alpha_2''}{\mathbf{case}_q(M?_r y_1.\alpha_1' \mid y_2.\alpha_2')\, \mathbf{of}\, x_1.\alpha_1 \mid x_2.\alpha_2 \dashrightarrow M?_r y_1.\alpha_1'' \mid y_2.\alpha_2''}$$

$$\frac{}{\mathbf{case}_q\, M{\downarrow}\, \mathbf{of}\, x_1.\alpha_1 \mid x_2.\alpha_2 \dashrightarrow M?_q x_1.\alpha_1 \mid x_2.\alpha_2}$$

5.5 Implication Introduction

$$\frac{M \hookrightarrow \alpha' \qquad \alpha' \hookrightarrow_S \alpha^{\bar{a}} \qquad \lambda_a x.\alpha^{\bar{a}} \dashrightarrow \alpha''}{\lambda_a x.M \hookrightarrow \alpha''}$$

$$\frac{\alpha^a \dashrightarrow_T M}{\lambda_a x.\alpha^a \dashrightarrow \lambda x.M{\downarrow}} \qquad \frac{}{\lambda_a x.M!_q \dashrightarrow M!_q} \qquad \frac{}{\lambda_a x.M \triangleright u \dashrightarrow M \triangleright u}$$

$$\frac{\lambda_a x.\alpha_1^{\bar{a}} \dashrightarrow \alpha_1' \qquad \lambda_a x.\alpha_2^{\bar{a}} \dashrightarrow \alpha_2'}{\lambda_a x.M?_q x_1.\alpha_1^{\bar{a}} \mid x_2.\alpha_2^{\bar{a}} \dashrightarrow M?_q x_1.\alpha_1' \mid x_2.\alpha_2'}$$

5.6 Implication Elimination

$$\frac{M_1 \hookrightarrow \alpha_1 \qquad M_2 \hookrightarrow \alpha_2 \qquad \alpha_1\ \alpha_2 \dashrightarrow \alpha'}{M_1\ M_2 \hookrightarrow \alpha'}$$

$$\frac{}{M_1{\downarrow}\ M_2{\downarrow} \dashrightarrow (M_1\ M_2){\downarrow}}$$

$$\frac{}{M!_q\ \alpha \dashrightarrow M!_q} \qquad \frac{}{\alpha\ M!_q \dashrightarrow M!_q}$$

$$\frac{}{(M \triangleright u)\ \alpha \dashrightarrow M \triangleright u} \qquad \frac{}{\alpha\ (M \triangleright u) \dashrightarrow M \triangleright u}$$

$$\frac{\alpha_1\ \alpha \dashrightarrow \alpha_1' \qquad \alpha_2\ \alpha \dashrightarrow \alpha_2'}{(M?_q x_1.\alpha_1 \mid x_2.\alpha_2)\ \alpha \dashrightarrow M?_q x_1.\alpha_1' \mid x_2.\alpha_2'}$$

$$\frac{\alpha\ \alpha_1 \dashrightarrow \alpha_1' \qquad \alpha\ \alpha_2 \dashrightarrow \alpha_2'}{\alpha\ (M?_q x_1.\alpha_1 \mid x_2.\alpha_2) \dashrightarrow M?_q x_1.\alpha_1' \mid x_2.\alpha_2'}$$

5.7 Truth Introduction

$$\frac{\langle\rangle \dashrightarrow \alpha'}{\langle\rangle \hookrightarrow \alpha'}$$

$$\frac{}{\langle\rangle \dashrightarrow \langle\rangle{\downarrow}}$$

5.8 Falsehood Elimination

$$\frac{M \hookrightarrow \alpha \qquad \mathbf{abort}_q\, \alpha \dashrightarrow \alpha'}{\mathbf{abort}_q\, M \hookrightarrow \alpha'}$$

$$\frac{}{\mathbf{abort}_q\, M! \dashrightarrow M!}$$

$$\frac{}{\mathbf{abort}_q(M \triangleright u) \dashrightarrow M \triangleright u}$$

$$\frac{\mathbf{abort}_q\, \alpha_1' \dashrightarrow \alpha_1'' \qquad \mathbf{abort}_q\, \alpha_2' \dashrightarrow \alpha_2''}{\mathbf{abort}_q(M?_r y_1.\alpha_1' \mid y_2.\alpha_2') \dashrightarrow M?_r y_1.\alpha_1'' \mid y_2.\alpha_2''}$$

$$\frac{}{\mathbf{abort}_q(M{\downarrow}) \dashrightarrow M!_q}$$

5.9 Throw

$$\frac{M \hookrightarrow \alpha \qquad \textbf{throw}\ \alpha\ \textbf{to}\ u \dashrightarrow \alpha'}{\textbf{throw}\ M\ \textbf{to}\ u \hookrightarrow \alpha'}$$

$$\frac{}{\textbf{throw}\ M!\ \textbf{to}\ u \dashrightarrow M!}$$

$$\frac{}{\textbf{throw}\ (M \triangleright v)\ \textbf{to}\ u \dashrightarrow M \triangleright v}$$

$$\frac{\textbf{throw}\ \alpha_1'\ \textbf{to}\ u \dashrightarrow \alpha_1'' \qquad \textbf{throw}\ \alpha_2'\ \textbf{to}\ u \dashrightarrow \alpha_2''}{\textbf{throw}\ (M?_r y_1.\alpha_1' \mid y_2.\alpha_2')\ \textbf{to}\ u \dashrightarrow M?_r y_1.\alpha_1'' \mid y_2.\alpha_2''}$$

$$\frac{}{\textbf{throw}\ (M\downarrow)\ \textbf{to}\ u \dashrightarrow M \triangleright u}$$

5.10 Letcc

$$\frac{M \hookrightarrow \alpha \qquad \textbf{letcc}\ u\ \textbf{in}\ \alpha \dashrightarrow \alpha'}{\textbf{letcc}\ u\ \textbf{in}\ M \hookrightarrow \alpha'}$$

$$\frac{}{\textbf{letcc}\ u\ \textbf{in}\ M! \dashrightarrow M!}$$

$$\frac{u \neq v}{\textbf{letcc}\ u\ \textbf{in}\ M \triangleright v \dashrightarrow M \triangleright v} \qquad \frac{}{\textbf{letcc}\ u\ \textbf{in}\ M \triangleright u \dashrightarrow M\downarrow}$$

$$\frac{\textbf{letcc}\ u\ \textbf{in}\ \alpha_1' \dashrightarrow \alpha_1'' \qquad \textbf{letcc}\ u\ \textbf{in}\ \alpha_2' \dashrightarrow \alpha_2''}{\textbf{letcc}\ u\ \textbf{in}\ M?_r y_1.\alpha_1' \mid y_2.\alpha_2' \dashrightarrow M?_r y_1.\alpha_1'' \mid y_2.\alpha_2''}$$

$$\frac{}{\textbf{letcc}\ u\ \textbf{in}\ M\downarrow \dashrightarrow M\downarrow}$$