Focused Linear Semantics of Modal Logic

Jason Reed

February 18, 2010

Abstract

The proof theory of the constructive modal logic of necessity and possibility described by Pfenning and Davies can be faithfully simulated in a first-order focused linear logic. This simulation sheds light on how the two modal operators $\Box$ and $\Diamond$ are related to one another, and how Pfenning-Davies' relates to other constructive accounts of modal logic.

1 Languages

We will set up three languages, BML, PML, FLL.

1.1 BML

BML is the *basic modal logic* which we take from Pfenning and Davies' judgmental reconstruction of modal logic. [PD01].

The syntax is

$$\begin{array}{rlll} \text{Propositions} & A & ::= & \bot \mid \top \mid A \vee A \mid A \wedge A \mid A \Rightarrow A \mid \Box A \mid \Diamond A \mid a \\ \text{Contexts} & \Gamma, \Delta & ::= & \cdot \mid \Gamma, A \\ \text{Conclusions} & J & ::= & A\,\mathsf{true} \mid A\,\mathsf{poss} \end{array}$$

The judgment is $\Delta; \Gamma \vdash J$, with Δ being the *valid context*, and Γ the *true context*. The proof system is in Figure 1. When the judgment on a conclusion is omitted, it is meant to be implicitly $A\,\mathsf{true}$.

1.2 PML

Next is the *polarized modal language* for which we only give a syntax and no proof theory. Its only role is to serve as a syntactic stepping-stone so that we can divide the translation from BML to FLL into two convenient phases. Its syntax is

$$\begin{array}{rlll} \text{Positives} & P & ::= & {\downarrow}N \mid \bot \mid P \vee P \mid \Box N \\ \text{Negatives} & N & ::= & {\uparrow}P \mid \top \mid N \wedge N \mid P \Rightarrow N \mid \Diamond P \mid a^- \end{array}$$

A proof system could however quite easily be given for PML, by pulling back the proof system for FLL along the translation from PML to FLL.

$$
\frac{\Delta;\Gamma, A \vdash B}{\Delta;\Gamma \vdash A \Rightarrow B}\Rightarrow R
\qquad
\frac{\Delta;\Gamma \vdash A \qquad \Delta;\Gamma, B \vdash J}{\Delta;\Gamma, A \Rightarrow B \vdash J}\Rightarrow L
\qquad
\frac{\Delta;\Gamma \vdash A \qquad \Delta;\Gamma \vdash B}{\Delta;\Gamma \vdash A \wedge B}\wedge R
$$

$$
\frac{\Delta;\Gamma, A_i \vdash J}{\Delta;\Gamma, A \wedge B \vdash J}\wedge L
\qquad
\frac{\Delta;\Gamma \vdash A_i}{\Delta;\Gamma \vdash A \vee B}\vee R_i
\qquad
\frac{\Delta;\Gamma, A \vdash J \qquad \Delta;\Gamma, B \vdash J}{\Delta;\Gamma, A \vee B \vdash J}\vee L
\qquad
\frac{}{\vdash \top}\top R
$$

$$
\frac{}{\Delta;\Gamma, \bot \vdash J}\bot L
\qquad
\frac{\Delta; \cdot \vdash A}{\Delta;\Gamma \vdash \Box A}\Box R
\qquad
\frac{\Delta, A;\Gamma \vdash J}{\Delta;\Gamma, \Box A \vdash J}\Box L
\qquad
\frac{\Delta;\Gamma, A \vdash J}{\Delta, A;\Gamma \vdash J}\mathit{copy}
$$

$$
\frac{}{\Delta;\Gamma, a \vdash a}\mathit{hyp}
\qquad
\frac{\Delta;\Gamma \vdash A\,\mathrm{poss}}{\Delta;\Gamma \vdash \Diamond A}\Diamond R
\qquad
\frac{\Delta; A \vdash C\,\mathrm{poss}}{\Delta;\Gamma, \Diamond A \vdash C\,\mathrm{poss}}\Diamond L
\qquad
\frac{\Delta;\Gamma \vdash A\,\mathrm{true}}{\Delta;\Gamma \vdash A\,\mathrm{poss}}\mathit{cocopy}
$$

Figure 1: BML Proof Rules

1.3 FLL

Here is first-order focused linear logic. Everything in here is fairly standard. One quirk is that we distinguish between 'packed' (propositional) atoms a^+ and a^- and 'unpacked' (judgmental) atoms a_+ and a_-. An 'unpacked' atom is what is left after we asynchronously perform the trivial decomposition of an atomic proposition, signaling that we need not decompose it any further. This is not really essential, but is a trick that makes proof-for-proof comparison of the translation with the original modal language go more easily.

The downshift $\downarrow N$ is missing only because we will not need it — since the original modal logic we're translating is unrestricted, everywhere $\downarrow$ would have been used, we use ! instead.

The syntax of first-order terms, propositions, contexts, and conclusions is as follows:

$$
\begin{array}{rccl}
\text{Terms} & q & ::= & \alpha \\
\text{Positives} & P & ::= & !N \mid \exists \alpha \geq q.P \mid a^+ \mid 0 \mid P \oplus P \mid P \otimes P \\
\text{Negatives} & N & ::= & \uparrow P \mid \forall \alpha \geq q.N \mid a^- \mid 1 \mid P \multimap N \\
\text{Linear Contexts} & \Gamma & ::= & \cdot \mid \Gamma, N \mid \Gamma, a_+ \\
\text{Valid Contexts} & \Delta & ::= & \cdot \mid \Delta, N \mid q_1 \geq q_2 \\
\text{Ordered Contexts} & \Omega & ::= & \cdot \mid \Omega, P \\
\text{Conclusions} & Q & ::= & P \mid a_-
\end{array}
$$

We understand contraction, exchange, and weakening to hold for Δ. We tacitly assume exchange for Γ. We have no structural rules for Ω apart from associativity.

The judgments are:

$$\frac{\Delta;\Gamma;\Omega,P \vdash_f N}{\Delta;\Gamma;\Omega \vdash_f P \multimap N}\multimap R \qquad \frac{\Delta;\Gamma_1 \vdash_f [P] \qquad \Delta;\Gamma_2[N] \vdash_f Q}{\Delta;\Gamma_1,\Gamma_2[P \multimap N] \vdash_f Q}\multimap L$$

$$\frac{\Delta;\Gamma_1 \vdash_f [P_1] \qquad \Delta;\Gamma_2 \vdash_f [P_2]}{\Delta;\Gamma_1,\Gamma_2 \vdash_f [P_1 \otimes P_2]}\otimes R \qquad \frac{\Delta;\Gamma;\Omega,P_1,P_2 \vdash_f Q}{\Delta;\Gamma;\Omega,P_1 \otimes P_2 \vdash_f Q}\otimes L \qquad \frac{\Delta;\Gamma \vdash_f [P_i]}{\Delta;\Gamma \vdash_f [P_1 \oplus P_2]}\oplus R_i$$

$$\frac{\Delta;\Gamma;\Omega,P_1 \vdash_f Q \qquad \Delta;\Gamma;\Omega,P_2 \vdash_f Q}{\Delta;\Gamma;\Omega,P_1 \oplus P_2 \vdash_f Q}\oplus L \qquad \frac{}{\Delta;\cdot \vdash_f 1}1R \qquad \frac{\Delta;\Gamma;\Omega \vdash_f Q}{\Delta;\Gamma;\Omega,1 \vdash_f Q}1L \qquad \frac{}{\Delta;\Gamma;\Omega,0 \vdash_f Q}0L$$

$$\frac{\Delta;\Gamma;\Omega \vdash_f P}{\Delta;\Gamma;\Omega \vdash_f \uparrow P}\uparrow R \qquad \frac{\Delta;\Gamma;P \vdash_f Q}{\Delta;\Gamma[\uparrow P] \vdash_f Q}\uparrow L \qquad \frac{\Delta;\Gamma;\cdot \vdash_f N}{\Delta;\Gamma \vdash_f [!N]}!R \qquad \frac{\Delta,N;\Gamma;\Omega \vdash_f Q}{\Delta;\Gamma;\Omega,!N \vdash_f Q}!L$$

$$\frac{}{\Delta;a_+ \vdash_f [a^+]}a^+R \qquad \frac{\Delta;\Gamma,a_+;\Omega \vdash_f Q}{\Delta;\Gamma;\Omega,a^+ \vdash_f Q}a^+L \qquad \frac{}{\Delta;\Gamma[a^-] \vdash_f a_-}a^-L \qquad \frac{\Delta;\Gamma;\Omega \vdash_f a_-}{\Delta;\Gamma;\Omega \vdash_f a^-}a^-R$$

$$\frac{\Delta;\Gamma \vdash_f [P]}{\Delta;\Gamma \vdash_f P}focR \qquad \frac{\Delta;\Gamma[N] \vdash_f P}{\Delta;\Gamma,N \vdash_f P}focL \qquad \frac{\Delta,N;\Gamma[N] \vdash_f P}{\Delta,N;\Gamma \vdash_f P}foc!L \qquad \frac{\Delta;\Gamma \vdash_f P}{\Delta;\Gamma;\cdot \vdash_f P}\cdot L$$

$$\frac{\Delta,\alpha \geq q;\Gamma;\Omega \vdash_f N(\alpha)}{\Delta;\Gamma;\Omega \vdash_f \forall\alpha \geq q.N(\alpha)}\forall R \qquad \frac{\Delta \vdash_f t \geq q \qquad \Delta;\Gamma[N(t)] \vdash_f Q}{\Delta;\Gamma[\forall\alpha \geq q.N(\alpha)] \vdash_f Q}\forall L$$

$$\frac{\Delta \vdash_f t \geq q \qquad \Delta;\Gamma \vdash_f [P(t)]}{\Delta;\Gamma \vdash_f [\exists\alpha \geq q.P(\alpha \geq q)]}\exists R \qquad \frac{\Delta,\alpha \geq q;\Gamma;\Omega,P(\alpha) \vdash_f Q}{\Delta;\Gamma;\Omega,\exists\alpha \geq q.P(\alpha) \vdash_f Q}\exists L$$

Figure 2: FLL Proof Rules

Stable	$\Delta;\Gamma \vdash_f Q$
Right Inversion	$\Delta;\Gamma;\Omega \vdash_f N$
Left Inversion	$\Delta;\Gamma;\Omega \vdash_f Q$
Right Focus	$\Delta;\Gamma \vdash_f [P]$
Left Focus	$\Delta;\Gamma[N] \vdash_f Q$

and the proof rules for the focusing system are in Figure 2. The f decorating the turnstile is merely to distinguish these judgments from the proof system for BML above.

Furthermore there is the judgment $\Delta \vdash q_1 \geq q_2$, which we take to mean that the abstract first-order relation $q_1 \geq q_2$ is deducible from assumptions of the same relation found in Δ, provided some axiomatization for properties of $\geq$ that is a parameter of the logic, just as frame properties are treated in Kripke semantics.

2 Translations

We will have one translation, the *polarization* from BML to PML, and another from PML to FLL, the *modal encoding.*

2.1 Polarization

Takes BML propositions and polarizes them 'maximally', so that even in a focusing calculus, proof search stops after every single BML connective decomposition. $\overleftarrow{A}$ is always negative, and $\overrightarrow{A}$ is always positive. Think of the arrow as pointing to where the proposition is synchronous and stable. Conversely, $\overset{\leftleftarrows}{A}$ is positive and $\overset{\rightrightarrows}{A}$ is always negative: for these translations, the arrow points to where the proposition is asynchronous and unstable.

$$\overset{\leftleftarrows}{A} = {\downarrow}\overleftarrow{A}$$
$$\overset{\rightrightarrows}{A} = {\uparrow}\overrightarrow{A}$$

A	$\overleftarrow{A}$	$\overrightarrow{A}$
$A_1 \vee A_2$	${\uparrow}(\overset{\leftleftarrows}{A}_1 \vee \overset{\leftleftarrows}{A}_2)$	${\downarrow}\overset{\rightrightarrows}{A}_1 \vee {\downarrow}\overset{\rightrightarrows}{A}_2$
$A_1 \wedge A_2$	${\uparrow}(\overset{\leftleftarrows}{A}_1 \wedge \overset{\leftleftarrows}{A}_2)$	${\downarrow}\overset{\rightrightarrows}{A}_1 \wedge {\downarrow}\overset{\rightrightarrows}{A}_2$
$A_1 \Rightarrow A_2$	${\downarrow}\overset{\rightrightarrows}{A}_1 \Rightarrow {\uparrow}\overset{\leftleftarrows}{A}_2$	${\downarrow}(\overset{\leftleftarrows}{A}_1 \Rightarrow \overset{\rightrightarrows}{A}_2)$
$\top$	${\uparrow}\top$	$\top$
$\bot$	${\uparrow}\bot$	$\bot$
$\Box A$	${\uparrow}\Box{\uparrow}\overset{\leftleftarrows}{A}$	$\Box\overset{\rightrightarrows}{A}$
$\Diamond A$	$\Diamond\overset{\leftleftarrows}{A}$	${\downarrow}\Diamond{\downarrow}\overset{\rightrightarrows}{A}$
a	a^-	${\downarrow}a^-$

2.2 Modal Encoding

Invent a positive atom $h^+(q)$ that expects one term argument, and a positive atom m^+ with no arguments.

Takes PML propositions and realizes the modal operations $\Box$ and $\Diamond$ as Kripke-like quantifiers over worlds. The central difference between Simpson's description [Sim94] of a constructive Kripke semantics of modal logic and ours is the linear token h^+ indexed by the current world that is required to be appropriately produced or consumed. This is essential to the 'context-clearing' effect of the judgmental $\Box$ and $\Diamond$ connectives. Furthermore, Pfenning-Davies $\Diamond$ has the requirement that the judgment poss already exist in the conclusion when $\Diamond L$ is invoked — this requirement is encoded by the extra $m^+ \multimap m^+ \otimes$ present at the beginning of the encoding of $\Diamond$, which reflects the essential *lax logical* character of this hypothesis-to-conclusion tethering effect, as the 'synthetic modality' $m^+ \multimap m^+ \otimes$ — by itself has the same proof theory as lax logic's $\bigcirc$.

The encoding function is as follows:

X	$X @ q$
$P_1 \vee P_2$	$(P_1 @ q) \oplus (P_2 @ q)$
$P_1 \wedge P_2$	$(P_1 @ q) \otimes (P_2 @ q)$
$P \Rightarrow N$	$(P @ q) \multimap (N @ q)$
$\top$	1
$\bot$	0
$\downarrow N$	$!(h^+(q) \multimap (N @ q))$
$\uparrow P$	$\uparrow(h^+(q) \otimes (P @ q))$
$\Box N$	$!(\forall \alpha \geq q.h^+(\alpha) \multimap (N @ \alpha))$
$\Diamond P$	$m^+ \multimap \uparrow(m^+ \otimes \exists \alpha \geq q.h^+(\alpha) \otimes (P @ \alpha))$
a^-	a^-

The correctness of these two translations chained together, in summary, is the following theorem. To prove it, the induction hypothesis must of course be strengthened to account for the effect of the translation on contexts.

Theorem 2.1 *If $\leq$ is axiomatized to be reflexive and transitive, then*

$$\vdash A \qquad \Leftrightarrow \qquad \vdash_f \overrightarrow{A} @ \alpha$$

References

[PD01] Frank Pfenning and Rowan Davies. A judgmental reconstruction of modal logic. *Mathematical Structures in Computer Science*, 11(4):511–540, 2001.

[Sim94] Alex K. Simpson. *The Proof Theory and Semantics of Intuitionistic Modal Logic*. PhD thesis, University of Edinburgh, 1994.