Logical Recipes I: Focusing

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Abstract

Exotic logics and logical phenomena abound: modal logics, substructural logics, constructive logics, classical logics; inversion, focusing, 'jumbo connectives', structural rules for logical judgments. But all of these can be cooked up out of intuitionistic first-order logic, while retaining isomorphic sets of proofs.

Focusing is the *sine qua non* of all of these encodings, the tool that allows careful control of proof construction. We begin by showing how it can be constructed out of an unfocused substrate.

1 Introduction

The purpose of this note is to show how to obtain a focusing [And92] proof theory for first-order logic via an encoding into ordinary, unfocused first-order logic. Previous work [Ree09] achieved this via translation into linear logic, but it turns out that it suffices to exploit the essential linearity of the intuitionistic conclusion.

The important metatheoretic properties of this focused proof theory — cut elimination, identity expansion, and the completeness of focusing — can be then proved *internally*, in a way that depends mostly on simple derivations in the target proof theory of the encoding.

2 Target Language

We start by describing the first-order intuitionistic logic we will be using as a target of the encoding. The language is largely standard:

$$\begin{array}{rrl} \text{Propositions} & A ::= & \exists x.A \mid A \wedge A \mid A \vee A \mid \\ & & \forall x.A \mid A \Rightarrow A \mid \top \mid \bot \mid a(\vec{t}) \mid \\ & & \mathsf{U}x.A \mid \mathsf{F}_t A \\ \text{Contexts} & \Gamma ::= & \cdot \mid \Gamma, A \end{array}$$

First-order terms are untyped, and propositional atoms a take a sequence $\vec{t}$ of first-order term arguments. We leave the exact set of atoms, and first-order

$$\frac{}{\Gamma, a \vdash a}\,\mathit{init}$$

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \Rightarrow B}\,{\Rightarrow}R \qquad \frac{\Gamma \vdash A \qquad \Gamma, B \vdash C}{\Gamma, A \Rightarrow B \vdash C}\,{\Rightarrow}L \qquad \frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \wedge B}\,{\wedge}R \qquad \frac{\Gamma, A, B \vdash C}{\Gamma, A \wedge B \vdash C}\,{\wedge}L$$

$$\frac{\Gamma \vdash A_i}{\Gamma \vdash A_1 \vee A_2}\,{\vee}R_i \qquad \frac{\Gamma, A \vdash C \qquad \Gamma, B \vdash C}{\Gamma, A \vee B \vdash C}\,{\vee}L \qquad \frac{}{\Gamma \vdash \top}\,{\top}R \qquad \frac{}{\Gamma, \bot \vdash C}\,{\bot}L$$

$$\frac{\Gamma \vdash A}{\Gamma \vdash \forall x.A}\,{\forall}R^x \qquad \frac{\Gamma, \{t/x\}A \vdash C}{\Gamma, \forall x.A \vdash C}\,{\forall}L \qquad \frac{\Gamma \vdash \{t/x\}A}{\Gamma \vdash \exists x.A}\,{\exists}R \qquad \frac{\Gamma, A \vdash C}{\Gamma, \exists x.A \vdash C}\,{\exists}L^x$$

$$\frac{\Gamma, A \vdash \#(x)}{\Gamma \vdash \mathsf{U}x.A}\,\mathsf{U}R^x \qquad \frac{\Gamma \vdash \{t/x\}A}{\Gamma, \mathsf{U}x.A \vdash \#(t)}\,\mathsf{U}L \qquad \frac{\Gamma, A \vdash \#(t)}{\Gamma \vdash \mathsf{F}_t A}\,\mathsf{F}_t R \qquad \frac{\Gamma \vdash A}{\Gamma, \mathsf{F}_t A \vdash \#(t)}\,\mathsf{F}_t L$$

Figure 1: Unfocused Proof Rules

function symbols unspecified. We sometimes leave the argument $\vec{t}$ of an atom implicit. All we will need is that there is a one-argument atom # that is distinct from any used by source-language propositions.

The proof rules are given in Figure 1. There are two slightly nonstandard connectives, a quantifier $\mathsf{U}x.A$ that binds a first-order variable x in A, and a unary operator $\mathsf{F}_t A$. We say that they are only 'slightly' nonstandard because it is easy to see that they are equivalent to aggregates of standard connectives in the sense that

$$\mathsf{U}x.A \dashv\vdash (A \Rightarrow \#(x)) \Rightarrow \#(x)$$
$$\mathsf{F}_t A \dashv\vdash A \Rightarrow \#(t)$$

So adding U and F to the first-order language is really no more unusual than adding negation $\neg$ and giving it its own inference rules, even though intuitionistic negation is definable in the sense that $\neg A \dashv\vdash A \Rightarrow \bot$. Indeed, $\mathsf{F}_t A$ resembles a kind of term-parametrized negation of A, and $\mathsf{U}x.A$ a quantified double-negation.

The essential results concerning this sequent calculus are the *identity* and *cut* theorems.

Theorem 2.1 (Identity) *$A \vdash A$ for any A.*

Theorem 2.2 (Cut) *If $\Gamma \vdash A$ and $\Gamma, A \vdash C$, then $\Gamma \vdash C$.*

These can be easily proved by standard methods. [Pfe94]

3 Source Language

We now describe the language of the focusing calculus that we encode into the above first-order logic. The language of propositions is *polarized* into positive and negative. There are *shift operators* $\uparrow$ and $\downarrow$ that coerce back and forth between the two polarities. We borrow symbols from linear logic to distinguish positive ($\otimes$) from negative ($\&$) conjunction, as well as the units (1 and $\top$, respectively) of those conjunctions.

$$
\begin{array}{rrcl}
\text{Positives} & P & ::= & \downarrow N \mid \exists x.P \mid P \otimes P \mid P \vee P \mid 1 \mid \bot \mid a^+(\vec{t}) \\
\text{Negatives} & N & ::= & \uparrow P \mid \forall x.N \mid P \Rightarrow N \mid N \;\&\; N \mid \top \mid a^-(\vec{t}) \\
\text{Positive Contexts} & \Omega & ::= & \cdot \mid P, \Omega \\
\text{Negative Contexts} & \Gamma & ::= & \cdot \mid \Gamma, N \mid \Gamma, \langle a^+(\vec{t}) \rangle \\
\text{Stable Conclusions} & Q & ::= & P \mid \langle a^-(\vec{t}) \rangle \\
\text{Conclusions} & R & ::= & N \mid Q
\end{array}
$$

The characteristic property of focusing is that when a proposition is asynchronous (positive on the left, negative on the right) it is eagerly inverted. If we reach an atom during this stage, it becomes a *suspension* $\langle a^-(\vec{t}) \rangle$ or $\langle a^+(\vec{t}) \rangle$. Inversion on the left takes place in a deterministic (but arbitrary) order, governed by an positive context Ω. Positive contexts are not subject to the usual structural properties of weakening, exchange, or contraction, but negative contexts Γ are. When all inversion is finished, we may choose a proposition to *focus* on, and as long as the focused proposition remains synchronous (negative on the left, positive on the right) we must remain focused on it.

The three judgments of the logic are

$$
\begin{array}{rl}
\text{Inversion} & \Gamma; \Omega \vdash_f R \\
\text{Right Focus} & \Gamma \vdash_f [P] \\
\text{Left Focus} & \Gamma[N] \vdash_f Q
\end{array}
$$

and the proof rules for the focusing system are in Figure 2. The f decorating the turnstile is merely to distinguish this logic from that of the target language.

We postpone stating any results about the focused logic, even though they can be proved directly, because our purpose is to prove them *through* the following translation.

4 Translation

The translation consists of four functions.

$N \bullet t$ takes a negative proposition and a term t to an unpolarized proposition.

N° takes a negative proposition to an unpolarized proposition.

$P^\bullet$ takes a positive proposition to an unpolarized proposition.

$P \circ A$ takes a positive proposition and an unpolarized proposition to an unpolarized proposition.

They are defined as follows:

$$\frac{\Gamma; P \vdash_f N}{\Gamma; \cdot \vdash_f P \Rightarrow N} \Rightarrow\!R \qquad \frac{\Gamma \vdash_f [P] \qquad \Gamma[N] \vdash_f Q}{\Gamma[P \Rightarrow N] \vdash_f Q} \Rightarrow\!L \qquad \frac{\Gamma; \cdot \vdash_f N_1 \qquad \Gamma; \cdot \vdash_f N_2}{\Gamma; \cdot \vdash_f N_1 \mathbin{\&} N_2} \mathbin{\&}R$$

$$\frac{\Gamma[N_i] \vdash_f Q}{\Gamma[N_1 \mathbin{\&} N_2] \vdash_f Q} \mathbin{\&}L \qquad \frac{\Gamma \vdash_f [P_1] \qquad \Gamma \vdash_f [P_2]}{\Gamma \vdash_f [P_1 \otimes P_2]} \otimes R \qquad \frac{\Gamma; P_1, P_2, \Omega \vdash_f R}{\Gamma; P_1 \otimes P_2, \Omega \vdash_f R} \otimes L$$

$$\frac{\Gamma \vdash_f [P_i]}{\Gamma \vdash_f [P_1 \vee P_2]} \vee R_i \qquad \frac{\Gamma; P_1, \Omega \vdash_f R \qquad \Gamma; P_2, \Omega \vdash_f R}{\Gamma; P_1 \vee P_2, \Omega \vdash_f R} \vee L \qquad \frac{}{\Gamma; \cdot \vdash_f \top} \top R \qquad \frac{}{\Gamma \vdash_f [1]} 1R$$

$$\frac{\Gamma; \Omega \vdash_f R}{\Gamma; 1, \Omega \vdash_f R} 1L \qquad \frac{}{\Gamma; \bot, \Omega \vdash_f R} \bot L \qquad \frac{\Gamma; \cdot \vdash_f N}{\Gamma; \cdot \vdash_f \forall x.N} \forall R^x \qquad \frac{\Gamma[\{t/x\}N] \vdash_f Q}{\Gamma[\forall x.N] \vdash_f Q} \forall L$$

$$\frac{\Gamma \vdash_f [\{t/x\}P]}{\Gamma \vdash_f [\exists x.P]} \exists R \qquad \frac{\Gamma, P; \Omega \vdash_f R}{\Gamma; \exists x.P, \Omega \vdash_f R} \exists L^x \qquad \frac{\Gamma; \cdot \vdash_f N}{\Gamma \vdash_f [{\downarrow}N]} {\downarrow}R \qquad \frac{\Gamma, N; \Omega \vdash_f R}{\Gamma; {\downarrow}N, \Omega \vdash_f R} {\downarrow}L$$

$$\frac{\Gamma; \cdot \vdash_f P}{\Gamma; \cdot \vdash_f {\uparrow}P} {\uparrow}R \qquad \frac{\Gamma; P \vdash_f Q}{\Gamma[{\uparrow}P] \vdash_f Q} {\uparrow}L \qquad \frac{}{\Gamma, \langle a^+ \rangle \vdash_f [a^+]} a^+R \qquad \frac{\Gamma, \langle a^+ \rangle; \Omega \vdash_f R}{\Gamma; a^+, \Omega \vdash_f R} a^+L$$

$$\frac{\Gamma; \cdot \vdash_f \langle a^- \rangle}{\Gamma; \cdot \vdash_f a^-} a^-R \qquad \frac{}{\Gamma[a^-] \vdash_f \langle a^- \rangle} a^-L \qquad \frac{\Gamma \vdash_f [P]}{\Gamma; \cdot \vdash_f P} focR \qquad \frac{\Gamma[N] \vdash_f Q}{\Gamma, N; \cdot \vdash_f Q} focL$$

Figure 2: Focusing Proof Rules

N	$N \bullet t$	N°
$P \Rightarrow N$	$P^\bullet \wedge N \bullet t$	$P \circ N^\circ$
$N_1 \mathbin{\&} N_2$	$N_1 \bullet t \vee N_2 \bullet t$	$N_1^\circ \wedge N_2^\circ$
$\top$	$\bot$	$\top$
$\uparrow P$	$P \circ \#(t)$	$\mathsf{U}x.\mathsf{F}_x P^\bullet$
$\forall x.N$	$\exists x.(N \bullet t)$	$\forall x.(N^\circ)$
$a^-(\vec{t})$	$a(\vec{t}, t)$	$\mathsf{U}x.a(\vec{t}, x)$

P	$P^\bullet$	$P \circ A$
$P_1 \otimes P_2$	$P_1^\bullet \wedge P_2^\bullet$	$P_1 \circ (P_2 \circ A)$
$P_1 \vee P_2$	$P_1^\bullet \vee P_2^\bullet$	$P_1 \circ A \wedge P_2 \circ A$
1	$\top$	A
$\bot$	$\bot$	$\top$
$\downarrow N$	N°	$(\mathsf{U}x.N \bullet x) \Rightarrow A$
$\exists x.P$	$\exists x.(P^\bullet)$	$\forall x.(P \circ A)$
$a^+(\vec{t})$	$a(\vec{t})$	$a(\vec{t}) \Rightarrow A$

We assume that we can choose a distinct target-language atom a for every source language atom a^+ and a^-.

Define $\Downarrow\Gamma$ and $\Uparrow_t Q$ by

$$\Downarrow\cdot = \cdot \qquad \Downarrow(\Gamma, N) = \Downarrow\Gamma, \mathsf{U}x.N \bullet x \qquad \Downarrow(\Gamma, \langle a^+(\vec{t})\rangle) = \Downarrow\Gamma, a(\vec{t})$$

$$\Uparrow_t P = \mathsf{F}_t P^\bullet \qquad \Uparrow_t \langle a^-(\vec{t})\rangle) = a(\vec{t}, t)$$

and $\Omega \circ A$ by

$$\cdot \circ A = A \qquad (P, \Omega) \circ A = P \circ (\Omega \circ A)$$

Say Ξ is a *garbage context for x* if it consists only of propositions $\Uparrow_t Q$ for $t \neq x$.

Theorem 4.1 (Correctness of Translation) *Suppose Ξ is a garbage context for x. Then the following pairs of sequents have isomorphic sets of proofs:*

$$\begin{array}{lll}
\textit{1.} & \Xi, \Downarrow\Gamma, \Uparrow_x Q \vdash \Omega \circ \#(x) & \cong \quad \Gamma; \Omega \vdash_f Q \\
\textit{2.} & \Xi, \Downarrow\Gamma \vdash P^\bullet & \cong \quad \Gamma \vdash_f [P] \\
\textit{3.} & \Xi, \Downarrow\Gamma, \Uparrow_x Q \vdash N \bullet x & \cong \quad \Gamma[N] \vdash_f Q \\
\textit{4.} & \Xi, \Downarrow\Gamma \vdash \Omega \circ N^\circ & \cong \quad \Gamma; \Omega \vdash_f N
\end{array}$$

Proof By induction on the relevant derivations. The key to the proof is that for anything in Ξ, $\Downarrow\Gamma$, or $\Uparrow_x Q$ to be usable, the conclusion must be a translated atom or of the form $\#(x)$, which can only happen in three circumstances:

1. in case 1, when Ω is empty, the conclusion can be $\#(x)$
2. in case 2, when $P = a^+(\vec{t})$, the conclusion can be $a(\vec{t})$.

3. in case 3, when $N = a^-(\vec{t})$, the conclusion can be $a(\vec{t}, t)$.

Outside of these special cases, we are forced to work on the right of the (target-language) sequent, which matches the constraints of the focusing system. Even then, nothing in Ξ is usable, because it was assumed to be garbage with respect to the term variable x. Decomposing a proposition in $\Downarrow\Gamma$ or $\Uparrow_x P$ in case 1 corresponds to a use of the rule $focL$ or $focR$, respectively. Working on $\Downarrow\langle a^+(\vec{t})\rangle = a(\vec{t})$ in case 2 or $\Uparrow_t\langle a^-(\vec{t})\rangle = a(\vec{t}, t)$ in case 3 corresponds to a use of $init$. ■

5 Metatheory of Focusing

For convenience we can define a further pair translation on positive and negative propositions like so:

N	N^*
$P \Rightarrow N$	$P^* \Rightarrow N^*$
$N_1 \mathbin{\&} N_2$	$N_1^* \wedge N_2^*$
$\top$	$\top$
$\uparrow P$	$\mathsf{U}x.\mathsf{F}_x P^*$
$\forall x.N$	$\forall x.(N^*)$
$a^-(\vec{t})$	$\mathsf{U}x.a(\vec{t}, x)$

P	P^*
$P_1 \otimes P_2$	$P_1^* \wedge P_2^*$
$P_1 \vee P_2$	$P_1^* \vee P_2^*$
1	$\top$
$\bot$	$\bot$
$\downarrow N$	N^*
$\exists x.P$	$\exists x.(P^*)$
$a^+(\vec{t})$	$a(\vec{t})$

These satisfy

Lemma 5.1

1. $\mathsf{U}x.N \bullet x \dashv\vdash N^*$

2. $N^\circ \dashv\vdash N^*$

3. $P^\bullet \dashv\vdash P^*$

4. $P \circ A \dashv\vdash P^* \Rightarrow A$

Proof By induction on the proposition. ■

Intuitively, the moral of the existence of the $*$ translation is that 'all that's really necessary for focusing' is that we interpret $\uparrow$ as a quantified double negation, and negative atoms as singly-negated.

Without doing any further inductions, we get identity and cut principles as follows:

Theorem 5.2 (Identity) $N \vdash_f N$ *and* $P \vdash_f P$

Proof By Theorem 4.1 it suffices to show $\mathsf{U}x.N \bullet x \vdash N^\circ$ and $\mathsf{F}_x P^\bullet \vdash P \circ \#(x)$. But these easily follow from Lemma 5.1. ■

Theorem 5.3 (Cut) *The following rules are admissible:*

$$\frac{\Gamma \vdash [P] \qquad \Gamma; P \vdash Q}{\Gamma \vdash Q} \qquad \frac{\Gamma; \cdot \vdash N \qquad \Gamma[N] \vdash Q}{\Gamma \vdash Q}$$

Proof By Theorem 4.1 it suffices to show the admissibility of

$$\frac{\Downarrow\Gamma \vdash P^\bullet \qquad \Downarrow\Gamma, \Uparrow_x Q \vdash P \circ \#(x)}{\Downarrow\Gamma, \Uparrow_x Q \vdash \#(x)} \qquad \frac{\Downarrow\Gamma \vdash N^\circ \qquad \Downarrow\Gamma, \Uparrow_x Q \vdash N \bullet x}{\Downarrow\Gamma, \Uparrow_x Q \vdash \#(x)}$$

But we can reason that

$$\dfrac{\Downarrow\Gamma \vdash P^\bullet \qquad \dfrac{\dfrac{\Downarrow\Gamma, \Uparrow_x Q \vdash P \circ \#(x) \qquad \dfrac{\text{Lemma 5.1}}{P \circ \#(x) \vdash \mathsf{F}_x P^\bullet}}{\Downarrow\Gamma, \Uparrow_x Q \vdash \mathsf{F}_x P^\bullet}\,cut \qquad \dfrac{[\text{easy proof}]}{\mathsf{F}_x P^\bullet, P^\bullet \vdash \#(x)}}{P^\bullet, \Downarrow\Gamma, \Uparrow_x Q \vdash \#(x)}\,cut}{\Downarrow\Gamma, \Uparrow_x Q \vdash \#(x)}\,cut$$

and

$$\dfrac{\dfrac{\Downarrow\Gamma \vdash N^\circ \quad \dfrac{\text{Lemma 5.1}}{N^\circ \vdash \mathsf{U}x.N \bullet x}}{\Downarrow\Gamma \vdash \mathsf{U}x.N \bullet x} \qquad \dfrac{\Downarrow\Gamma, \Uparrow_x Q \vdash N \bullet x}{\Downarrow\Gamma, \Uparrow_x Q, \mathsf{U}x.N \bullet x \vdash \#(x)}\,\mathsf{U}L}{\Downarrow\Gamma, \Uparrow_x Q \vdash \#(x)}\,cut$$

■

The completeness of focusing essentially amounts the ability to eliminate double-shifts without affecting provability. Let $X^\Box$ (where X is either P or N) be X with all instances of $\downarrow\uparrow$ or $\uparrow\downarrow$ removed. For convenience in proving the following lemmas, abbreviate $\bigcirc A = \mathsf{U}x.\mathsf{F}_x A$ and let $\Gamma \vdash A$ lax stand for $\Gamma, \mathsf{F}_x A \vdash \#(x)$. Notice that all the usual rules of lax logic are admissible, i.e.

$$\frac{\Gamma \vdash A \text{ lax}}{\Gamma \vdash \bigcirc A} \qquad \frac{\Gamma \vdash A}{\Gamma \vdash A \text{ lax}} \qquad \frac{\Gamma, A \vdash C \text{ lax}}{\Gamma, \bigcirc A \vdash C \text{ lax}}$$

Here are some more easy lemmas about $\bigcirc$:

Lemma 5.4

1. $\mathsf{F}_t \bigcirc A \dashv\vdash \mathsf{F}_t A$
2. $(\mathsf{U}x.A) \dashv\vdash \bigcirc(\mathsf{U}x.A)$
3. $N^* \dashv\vdash \bigcirc N^*$

Proof

1. To prove $\mathsf{F}_t \bigcirc A \vdash \mathsf{F}_t A$, we reason that

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash A}}{A \vdash A \ \mathrm{lax}}}{A \vdash \bigcirc A}}{\mathsf{F}_t \bigcirc A, A \vdash \#(t)}}{\mathsf{F}_t \bigcirc A \vdash \mathsf{F}_t A}$$

   The other direction is

$$\dfrac{\dfrac{\dfrac{}{\mathsf{F}_t A \vdash \mathsf{F}_t A}}{\mathsf{F}_t A, \bigcirc A \vdash \#(t)}}{\mathsf{F}_t A \vdash \mathsf{F}_t \bigcirc A}$$

2. The $(\mathsf{U}x.A) \vdash \bigcirc(\mathsf{U}x.A)$ direction is trivial. The other direction is

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash A}}{A, \mathsf{U}x.A \vdash \#(x)}}{A \vdash \mathsf{F}_x \mathsf{U}x.A}}{\bigcirc(\mathsf{U}x.A), A \vdash \#(x)}}{\bigcirc(\mathsf{U}x.A) \vdash \mathsf{U}x.A}$$

3. $N^* \dashv\vdash \mathsf{U}x.N \bullet x \Rightarrow \#(x) \dashv\vdash \bigcirc \mathsf{U}x.N \bullet x \Rightarrow \#(x) \dashv\vdash \bigcirc N^*$.

■

Theorem 5.5 (Completeness of Focusing)

1. $N^{\square *} \dashv\vdash N^*$
2. $\bigcirc P^{\square *} \dashv\vdash \bigcirc P^*$

Proof

1. If $N = \uparrow\downarrow N_0$, we must show $N_0^{\square *} \dashv\vdash \bigcirc N_0^*$. In this case, appeal to the induction hypothesis to see that $N_0^{\square *} \dashv\vdash N_0^*$ and Lemma 5.4 (part 3) to see that $N_0^* \dashv\vdash \bigcirc N_0^*$. Otherwise, N is some propositional connective. We split cases.

   Case: $P \Rightarrow N$. Reason that

$P^{\square *} \Rightarrow N^{\square *} \dashv\vdash P^{\square *} \Rightarrow N^*$	i.h.
$\dashv\vdash P^{\square *} \Rightarrow \mathsf{U}x.N \bullet x \Rightarrow \#(x)$	Lemma 5.1
$\dashv\vdash \mathsf{U}x.N \bullet x \Rightarrow P^{\square *} \Rightarrow \#(x)$	

$\dashv\vdash \mathsf{U}x.N \bullet x \Rightarrow \mathsf{F}_x P^{\Box *}$
$\dashv\vdash \mathsf{U}x.N \bullet x \Rightarrow \mathsf{F}_x \bigcirc P^{\Box *}$ Lemma 5.4 (1)
$\dashv\vdash \mathsf{U}x.N \bullet x \Rightarrow \mathsf{F}_x \bigcirc P^{*}$ i.h.
$\dashv\vdash \mathsf{U}x.N \bullet x \Rightarrow \mathsf{F}_x P^{*}$ Lemma 5.4 (1)
$\dashv\vdash \mathsf{U}x.N \bullet x \Rightarrow P^{*} \Rightarrow \#(x)$
$\dashv\vdash P^{*} \Rightarrow \mathsf{U}x.N \bullet x \Rightarrow \#(x)$
$\dashv\vdash P^{*} \Rightarrow N^{*}$ Lemma 5.1

Case: $\uparrow P$. We must show $\bigcirc P^{\Box *} \dashv\vdash \bigcirc P^{*}$, but this follows immediately from the induction hypothesis.

Case: $N_1 \,\&\, N_2$. We must show $N_1^{\Box *} \,\&\, N_2^{\Box *} \dashv\vdash N_1^{*} \,\&\, N_2^{*}$ but this follows easily from the induction hypothesis.

Case: $\forall x.N$. We must show $\forall x.N^{\Box *} \dashv\vdash \forall x.N^{*}$ but this follows easily from the induction hypothesis.

Case: $a^{-}(\vec{t})$. In this case $N^{\Box} = N$ and we are done.

Case: $\top$. In this case $N^{\Box} = N$ and we are done.

2. If $P = \downarrow\uparrow P_0$, we must show $\bigcirc P_0^{\Box *} \dashv\vdash \bigcirc\bigcirc P_0^{*}$. By the induction hypothesis, we know $\bigcirc P_0^{\Box *} \dashv\vdash \bigcirc P_0^{*}$, and showing that $\bigcirc P_0^{*} \dashv\vdash \bigcirc\bigcirc P_0^{*}$ is an easy tautology of lax logic. Otherwise we split cases on P.

Case: $P_1 \otimes P_2$. Observe that $\bigcirc(A \wedge B) \dashv\vdash \bigcirc(\bigcirc A \wedge \bigcirc B)$. We reason that

$\bigcirc(P_1^{\Box *} \wedge P_2^{\Box *})$
$\dashv\vdash \bigcirc(\bigcirc P_1^{\Box *} \wedge \bigcirc P_2^{\Box *})$ observation
$\dashv\vdash \bigcirc(\bigcirc P_1^{*} \wedge \bigcirc P_2^{*})$ i.h.
$\dashv\vdash \bigcirc(P_1^{*} \wedge P_2^{*})$ observation

Case: $P_1 \vee P_2$. Observe that $\bigcirc(A \vee B) \dashv\vdash \bigcirc(\bigcirc A \vee \bigcirc B)$. We reason that

$\bigcirc(P_1^{\Box *} \vee P_2^{\Box *})$
$\dashv\vdash \bigcirc(\bigcirc P_1^{\Box *} \vee \bigcirc P_2^{\Box *})$ observation
$\dashv\vdash \bigcirc(\bigcirc P_1^{*} \vee \bigcirc P_2^{*})$ i.h.
$\dashv\vdash \bigcirc(P_1^{*} \vee P_2^{*})$ observation

Case: $\exists x.P$. Observe that $\bigcirc\exists x.A \dashv\vdash \bigcirc(\exists x.\bigcirc A)$. We reason that

$\bigcirc(\exists x.P^{\Box *})$
$\dashv\vdash \bigcirc(\exists x.\bigcirc P^{\Box *})$ observation
$\dashv\vdash \bigcirc(\exists x.\bigcirc P^{*})$ i.h.
$\dashv\vdash \bigcirc(\exists x.P^{*})$ observation

Case: $a^{+}(\vec{t})$. In this case $P^{\Box} = P$ and we are done.

Case: 1. In this case $P^{\Box} = P$ and we are done.

Case: $\bot$. In this case $P^{\Box} = P$ and we are done.

Case: $\downarrow N$. We must show $\bigcirc N^{\Box *} \dashv\vdash \bigcirc N^*$, but this follows immediately from the induction hypothesis.

■

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