Logical Recipes II: Linear Logic

Jason Reed

June 27, 2013

Abstract

Linear logic has significant structure in it contexts, relative to ordinary intuitionistic logic. Assumptions cannot be freely duplicated or dropped, but are managed in a multiset. We show how to reduce its context structure to a data structure at the term level in first-order logic. This is focusing-compatible: a focusing proof theory for linear logic can be read off from the translation.

1 Introduction

The purpose of this note is to show how to obtain a focusing [And92] proof theory for linear logic via an encoding into focused first-order logic.

Two important metatheoretic properties of the proof theory for linear logic — cut elimination, identity expansion — come for free because the translation simply maps propositions to propositions uniformly. The completeness of focusing for linear logic can be then proved *internally*, in a way that depends mostly on simple derivations in the target proof theory of the encoding.

2 Target Language

The target language of this recipe is the source language of the previous one, namely focused first-order intuitionistic logic. We list the language of its propositions (apart from atoms) here, and refer the reader to [Ree13] for its proof theory.

$$\begin{array}{rccl} \text{Positives} & \bar{P} & ::= & {\downarrow}N \mid \exists x.\bar{P} \mid \bar{P} \otimes \bar{P} \mid \bar{P} \vee \bar{P} \mid 1 \mid \bot \\ \text{Negatives} & \bar{N} & ::= & {\uparrow}\bar{P} \mid \forall x.\bar{N} \mid \bar{P} \Rightarrow \bar{N} \mid \bar{N} \mathbin{\&} \bar{N} \mid \top \end{array}$$

We write bars over all propositional metavariables just to distinguish them from the P and N of the source language below.

To represent linear logic, we will break up the untyped domain t of first-order terms into a few distinct sorts. There will be *resources*, *frames*, and *structures*. A structure represents the shape of a linear logic sequent. A resource represents the shape of a sequent minus its conclusion, in other words, a context. A frame represents the shape of a sequent minus one hypothesis, in other words, a context and a conclusion.

$$\begin{array}{rccl} \text{Resources} & r & ::= & \rho \mid r \otimes r \mid 1 \\ \text{Frames} & f & ::= & \phi \mid r \multimap f \\ \text{Structures} & s & ::= & r \rhd f \end{array}$$

We write ρ for resource variables and ϕ for frame variables. We will have no need for structure variables. The reuse of linear logic symbols $\otimes, 1, \multimap$ is intentional and meant to be suggestive. We will see that resources 'go with' positive propositions, and frames 'go with' negatives in such a way that this use of these symbols in connection with the propositional connectives they judgmentally internalize makes sense.

We also allow equations between first-order terms in the target language, and impose the following equational theory on resources, frames, and structures:

$$r_1 \otimes (r_2 \otimes r_3) = (r_1 \otimes r_2) \otimes r_3$$

$$r_1 \otimes r_2 = r_2 \otimes r_1$$

$$r \otimes 1 = 1$$

$$r_1 \otimes r_2 \rhd f = r_1 \rhd r_2 \multimap f$$

This appears to be cheating a little, since we said nothing about equality in the previous recipe. Notionally what we are doing is adducing $=$ as a negative atomic formula on two terms, in which case the axioms become mere hypotheses in an extra globally present context. The reason we choose negative polarity is so that we can see that the axioms can only be used when the conclusion is alread an equality. This isolates equational reasoning from the rest of proof search. Technically, to obtain true focal adequacy with respect to linear logic, we would need to impose proof-irrelevance as soon as we invert an axiom $t = t$ on the right. In the sequel, however, we will freely use the equations as if they are definitional equalities.

Note that because of these equations, every structure expression can be normalized into the form

$$\rho_1 \otimes \cdots \otimes \rho_n \rhd \phi$$

where the parenthesization and order of the ρs is immaterial. This is the shape of a linear logic context: a collection of hypotheses and a conclusion.

We require one negative atom $\#^-(s)$ which, when it occurs in the conclusion of the target language sequent, represents the *current* linear logic sequent. However, since s has a unique top-level function symbol $\rhd$, and since such atoms occur so frequently, we just write $r \rhd f$ instead of $\#^-(r \rhd f)$.

We require one positive atom $b^+(a^-, f)$ for each source-language negative atom, and one positive atom $b^+(a^+, r)$ for each source-language positive atom. Yes, you read correctly: the target-language atom b^+ is positive regardless of the polarity of the source-language atom, although the sort of its second argument differs. (cf. the above comment that frames go with negative and resources go with positive)

3 Source Language

We now describe focused intuitionistic linear logic. As always with focusing proof theories, the language of propositions is *polarized* into positive and negative. There are again shift operators $\uparrow$ and $\downarrow$ that coerce back and forth between the two polarities.

$$
\begin{array}{rcll}
\text{Positives} & P & ::= & \downarrow N \mid P \otimes P \mid P \oplus P \mid 1 \mid 0 \mid a^+ \mid !N \\
\text{Negatives} & N & ::= & \uparrow P \mid P \multimap N \mid N \mathbin{\&} N \mid \top \mid a^- \\
\text{Positive Contexts} & \Omega & ::= & \cdot \mid P, \Omega \\
\text{Negative Contexts} & \Gamma & ::= & \cdot \mid \Gamma, N \mid \Gamma, N \text{ valid} \mid \Gamma, \langle a^+ \rangle \\
\text{Linear Hypotheses} & H & ::= & N \mid \langle a^+ \rangle \\
\text{Stable Conclusions} & Q & ::= & P \mid \langle a^- \rangle \\
\text{Conclusions} & R & ::= & N \mid Q
\end{array}
$$

The three judgments of the logic are

$$
\begin{array}{rl}
\text{Inversion} & \Gamma; \Omega \vdash_{\ell} R \\
\text{Right Focus} & \Gamma \vdash_{\ell} [P] \\
\text{Left Focus} & \Gamma[N] \vdash_{\ell} Q
\end{array}
$$

and the proof rules for the focusing system are in Figure 1. The ℓ decorating the turnstile is merely to distinguish this logic from that of the target language. We write Δ for a Γ that consists only of N valid hypotheses. These N valid hypotheses are allowed to be subject to weakening and contraction; linear hypotheses N are not.

4 Translation

We make the abbreviations

- $\mathsf{U}^f \rho.\bar{P} = \forall \rho.\bar{P} \Rightarrow \rho \triangleright f$ and
- $\mathsf{U}^r \phi.\bar{P} = \forall \phi.\bar{P} \Rightarrow r \triangleright \phi$.

The core of the translation consists of two functions. N^f takes a negative proposition and a frame to a target-language positive proposition. P^r takes a positive proposition and a resource to a target-language positive proposition. They are defined in Figure 2.

Here are a few further definitions so that we can state the adequacy of the translation: For any vector $\vec{r}$ of resources, define $\Downarrow_{\vec{r}}\Gamma$

$$\Downarrow_{\cdot} \cdot = \cdot \qquad \Downarrow_{\vec{r},r}(\Gamma, H) = \Downarrow_{\vec{r}}\Gamma, (\Downarrow_r H)$$

and $\Downarrow_r H$ by

$$\Downarrow_r N = \mathsf{U}^r \phi.N^{\phi} \qquad \Downarrow_r \langle a^+ \rangle = \langle b^+(a^+, r) \rangle$$

Define $\Downarrow\Delta$ and $\Uparrow_f Q$ by

$$\Downarrow \cdot = \cdot \qquad \Downarrow(\Delta, N \text{ valid}) = \Downarrow\Delta, \mathsf{U}^1 \phi.N^{\phi}$$

$$\frac{\Gamma; P \vdash_\ell N}{\Gamma; \cdot \vdash_\ell P \multimap N} \multimap R \qquad \frac{\Gamma_1 \vdash_\ell [P] \qquad \Gamma_2[N] \vdash_\ell Q}{\Gamma_1, \Gamma_2[P \multimap N] \vdash_\ell Q} \multimap L \qquad \frac{\Gamma; \cdot \vdash_\ell N_1 \qquad \Gamma; \cdot \vdash_\ell N_2}{\Gamma; \cdot \vdash_\ell N_1 \,\&\, N_2} \& R$$

$$\frac{\Gamma[N_i] \vdash_\ell Q}{\Gamma[N_1 \,\&\, N_2] \vdash_\ell Q} \& L \qquad \frac{\Gamma_1 \vdash_\ell [P_1] \qquad \Gamma_2 \vdash_\ell [P_2]}{\Gamma_1, \Gamma_2 \vdash_\ell [P_1 \otimes P_2]} \otimes R \qquad \frac{\Gamma; P_1, P_2, \Omega \vdash_\ell R}{\Gamma; P_1 \otimes P_2, \Omega \vdash_\ell R} \otimes L$$

$$\frac{\Gamma \vdash_\ell [P_i]}{\Gamma \vdash_\ell [P_1 \oplus P_2]} \oplus R_i \qquad \frac{\Gamma; P_1, \Omega \vdash_\ell R \qquad \Gamma; P_2, \Omega \vdash_\ell R}{\Gamma; P_1 \oplus P_2, \Omega \vdash_\ell R} \oplus L \qquad \frac{}{\Gamma; \cdot \vdash_\ell \top} \top R \qquad \frac{}{\Delta \vdash_\ell [1]} 1R$$

$$\frac{\Gamma; \Omega \vdash_\ell R}{\Gamma; 1, \Omega \vdash_\ell R} 1L \qquad \frac{}{\Gamma; 0, \Omega \vdash_\ell R} 0L \qquad \frac{\Gamma; \cdot \vdash_\ell N}{\Gamma \vdash_\ell [{\downarrow}N]} {\downarrow}R \qquad \frac{\Gamma, N; \Omega \vdash_\ell R}{\Gamma; {\downarrow}N, \Omega \vdash_\ell R} {\downarrow}L \qquad \frac{\Gamma; \cdot \vdash_\ell P}{\Gamma; \cdot \vdash_\ell {\uparrow}P} {\uparrow}R$$

$$\frac{\Gamma; P \vdash_\ell Q}{\Gamma[{\uparrow}P] \vdash_\ell Q} {\uparrow}L \qquad \frac{}{\Gamma, \langle a^+ \rangle \vdash_\ell [a^+]} a^+R \qquad \frac{\Gamma, \langle a^+ \rangle; \Omega \vdash_\ell R}{\Gamma; a^+, \Omega \vdash_\ell R} a^+L \qquad \frac{\Gamma; \cdot \vdash_\ell \langle a^- \rangle}{\Gamma; \cdot \vdash_\ell a^-} a^-R$$

$$\frac{}{\Gamma[a^-] \vdash_\ell \langle a^- \rangle} a^-L \qquad \frac{\Gamma \vdash_\ell [P]}{\Gamma; \cdot \vdash_\ell P} focR \qquad \frac{\Gamma[N] \vdash_\ell Q}{\Gamma, N; \cdot \vdash_\ell Q} focL$$

$$\frac{\Gamma[N] \vdash_\ell Q}{\Gamma, N \text{ valid}; \cdot \vdash_\ell Q} foc! \qquad \frac{\Delta; \cdot \vdash_\ell N}{\Delta \vdash_\ell [!N]} !R \qquad \frac{\Gamma, N \text{ valid}; \Omega \vdash_\ell R}{\Gamma; !N, \Omega \vdash_\ell R} !L$$

Figure 1: Focused Linear Logic Proof Rules

N	N^f
$P \multimap N$	$\exists\rho.P^\rho \wedge \exists\phi.N^\phi \wedge f = \rho \multimap \phi$
$N_1 \,\&\, N_2$	$N_1^f \vee N_2^f$
$\top$	0
${\uparrow}P$	${\downarrow}\mathsf{U}^f \rho.P^\rho$
a^-	$b^+(a^-, f)$

P	P^r
$P_1 \otimes P_2$	$\exists\rho_1.P_1^{\rho_1} \wedge \exists\rho_2.P_2^{\rho_2} \wedge r = \rho_1 \otimes \rho_2$
$P_1 \oplus P_2$	$P_1^r \vee P_2^r$
1	$r = 1$
0	0
${\downarrow}N$	${\downarrow}\mathsf{U}^r \phi.N^\phi$
$!N$	$r = 1 \wedge {\downarrow}\mathsf{U}^1 \phi.N^\phi$
a^+	$b^+(a^+, r)$

Figure 2: Translation

and

$$\Uparrow_f P = \mathsf{U}^f \rho . P^\rho \qquad \Uparrow_f \langle a^- \rangle = \langle b^+(a^-, f) \rangle$$

Define $\Omega^{\vec{r}}$ by

$$\cdot^{\cdot} = \cdot \qquad (P, \Omega) \cdot (r, \vec{r}) = P^r, \Omega^{\vec{r}}$$

Say Ξ is a *garbage context for s* if it consists only of propositions $\Uparrow_\phi Q$ and $\Downarrow_\rho H$ for $\phi \notin s$ and $\rho \notin s$.

When $\vec{\rho} = (\rho_1, \ldots, \rho_n)$, we write $\otimes(\vec{\rho})$ for $\rho_1 \otimes \cdots \otimes \rho_n$.

Theorem 4.1 (Correctness of Translation) *Suppose Ξ is a garbage context for x. Then the following pairs of sequents have isomorphic sets of proofs:*

$$\begin{array}{lllll}
\mathit{1.} & \Xi, \Downarrow\Delta, \Downarrow_{\vec{\rho}}\Gamma, \Uparrow_\phi Q; \Omega^{\vec{\rho}_0} \vdash \#(\otimes(\vec{\rho}\vec{\rho}_0) \rhd \phi)) & \cong & \Gamma; \Omega \vdash_\ell Q \\
\mathit{2.} & \Xi, \Downarrow\Delta, \Downarrow_{\vec{\rho}}\Gamma; \Omega^{\vec{\rho}_0}, N^\phi \vdash \#(\otimes(\vec{\rho}\vec{\rho}_0) \rhd \phi) & \cong & \Gamma; \Omega \vdash_\ell N \\
\mathit{3.} & \Xi, \Downarrow\Delta, \Downarrow_{\vec{\rho}}\Gamma, \Uparrow_\phi Q \vdash [N^{\otimes(\vec{\rho}) \multimap \phi}] & \cong & \Gamma[N] \vdash_\ell Q \\
\mathit{4.} & \Xi, \Downarrow\Delta, \Downarrow_{\vec{\rho}}\Gamma \vdash [P^{\otimes(\vec{\rho})}] & \cong & \Gamma \vdash_\ell [P]
\end{array}$$

Proof By induction on the relevant derivations. The key to the proof is that in nearly every case, focusing tightly constrains both sides of the isomorphism. The only case where we get to choose what to work on is case 1 when Ω is empty. In this case, we must show

$$\Xi, \Downarrow\Delta, \Downarrow_{\vec{\rho}}\Gamma, \Uparrow_\phi Q; \cdot \vdash \#(\otimes(\vec{\rho}) \rhd \phi)) \qquad \cong \qquad \Gamma; \cdot \vdash_\ell Q$$

Decomposing a proposition in $\Downarrow\Delta$ or $\Downarrow_{\vec{\rho}}\Gamma$ or $\Uparrow_\phi Q$ corresponds to a use of the rule $foc!$ or $focL$ or $focR$, respectively. It is easy to see that any attempt to focus on a proposition in Ξ will meet with immediate failure, by reasoning about the definition of garbage context. ∎

5 Metatheory of Focusing

Since this translation is 'one-sided' (in the sense that every linear logic proposition maps down to a single focused first-order logic proposition) rather than the 'two-sided' translation we used to encode focusing itself (where the translation depended on whether the source proposition occurred in a negative or positive position), the identity and cut theorems for linear logic are inherited directly from the same theorems for the focused logic. It remains to show that focusing is complete for linear logic.

The completeness of focusing again essentially amounts to the ability to eliminate double-shifts without affecting provability. Let $X^\Box$ (where X is either P or N) be X with all instances of $\downarrow\uparrow$ or $\uparrow\downarrow$ removed.

Note that in the target language we can easily prove

Lemma 5.1 (Brouwer's Lemma)

1. $\mathsf{U}^f \rho . \downarrow \mathsf{U}^\rho \phi . \downarrow \mathsf{U}^\phi \rho' . \bar{P} \dashv\vdash \mathsf{U}^f \rho' . P$.

2. $\mathsf{U}^r\phi.{\downarrow}\mathsf{U}^\phi\rho.{\downarrow}\mathsf{U}^\rho\phi'.\bar{P} \dashv\vdash \mathsf{U}^r\phi'.P$.

With this lemma it is more or less straightforward to obtain

Theorem 5.2 (Completeness of Focusing) *For any r, f, we have*

1. $\mathsf{U}^r\phi.N^{\Box\phi} \dashv\vdash \mathsf{U}^r\phi.N^\phi$
2. $\mathsf{U}^f\rho.P^{\Box\rho} \dashv\vdash \mathsf{U}^f\rho.P^\rho$

Proof If $N = {\uparrow}{\downarrow}N_0$ or $P = {\downarrow}{\uparrow}P_0$, apply Brouwer's Lemma. Otherwise we split cases on the proposition. The most interesting cases are the multiplicatives $\multimap$ and $\otimes$. We show the case for $\multimap$.

Here we must show

$$\mathsf{U}^r\phi.\exists\rho.P^{\Box\rho} \wedge \exists\phi_0.N^{\Box\phi_0} \wedge \phi = \rho \multimap \phi_0 \dashv\vdash \mathsf{U}^r\phi.\exists\rho.P^\rho \wedge \exists\phi_0.N^{\phi_0} \wedge \phi = \rho \multimap \phi_0]$$

We do this by reasoning in the target language. The adjunction axiom $r_1 \otimes r_2 \triangleright f = r_1 \triangleright r_2 \multimap f$ turns out to be crucial to our ability to apply the induction hypothesis.

We reason like so:

$$\begin{array}{lr}
\quad \mathsf{U}^r\phi.(\exists\rho.P^{\Box\rho} \wedge \exists\phi_0.N^{\Box\phi_0} \wedge \phi = \rho \multimap \phi_0) & \\
\dashv\vdash \forall\rho.P^{\Box\rho} \Rightarrow \forall\phi_0.N^{\Box\phi_0} \Rightarrow r \triangleright (\rho \multimap \phi_0) & \\
\dashv\vdash \forall\rho.P^{\Box\rho} \Rightarrow \forall\phi_0.N^{\Box\phi_0} \Rightarrow r \otimes \rho \triangleright \phi_0 & \text{adjunction} \\
\dashv\vdash \forall\rho.P^{\Box\rho} \Rightarrow \mathsf{U}^{r\otimes\rho}\phi_0.N^{\Box\phi_0} & \\
\dashv\vdash \forall\rho.P^{\Box\rho} \Rightarrow \mathsf{U}^{r\otimes\rho}\phi_0.N^{\phi_0} & \text{i.h.} \\
\dashv\vdash \forall\rho.P^{\Box\rho} \Rightarrow \forall\phi_0.N^{\phi_0} \Rightarrow r \otimes \rho \triangleright \phi_0 & \\
\dashv\vdash \forall\rho.P^{\Box\rho} \Rightarrow \forall\phi_0.N^{\phi_0} \Rightarrow \rho \triangleright r \multimap \phi_0 & \text{adjunction} \\
\dashv\vdash \forall\phi_0.N^{\phi_0} \Rightarrow \forall\rho.P^{\Box\rho} \Rightarrow \rho \triangleright r \multimap \phi_0 & \\
\dashv\vdash \forall\phi_0.N^{\phi_0} \Rightarrow \mathsf{U}^{r\multimap\phi_0}\rho.P^{\Box\rho} & \\
\dashv\vdash \forall\phi_0.N^{\phi_0} \Rightarrow \mathsf{U}^{r\multimap\phi_0}\rho.P^{\rho} & \text{i.h.} \\
\dashv\vdash \forall\phi_0.N^{\phi_0} \Rightarrow \forall\rho.P^{\rho} \Rightarrow \rho \triangleright r \multimap \phi_0 & \\
\dashv\vdash \forall\phi_0.N^{\phi_0} \Rightarrow \forall\rho.P^{\rho} \Rightarrow r \triangleright \rho \multimap \phi_0 & \text{adjunction} \\
\dashv\vdash \mathsf{U}^r\phi.(\exists\rho.P^\rho \wedge \exists\phi_0.N^{\phi_0} \wedge \phi = \rho \multimap \phi_0) &
\end{array}$$

■

References

[And92] J. M. Andreoli. Logic programming with focusing proofs in linear logic. *Journal of Logic and Computation*, 2(3):297–347, 1992.

[Ree13] Jason Reed. Logical recipes I: Focusing. Unpublished manuscript, 2013.