Notes on M&M Chapter IV

Jason Reed

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1 Elementary Topos Theory

1.1 Topoi

An *elementary topos* (or just a topos) is a category with pullbacks, a terminal object, a subobject classifier, and to every object A a power object PA such that for every arrow $f : B \times A \to \Omega$ there is a unique arrow $\bar{f} : A \to PB$ such that

$$\begin{tikzcd}
B \times A \arrow[r, "f"] \arrow[d, dashed, "B \times \bar{f}"'] & \Omega \arrow[d, equal] \\
B \times PB \arrow[r, "\in_B"'] & \Omega
\end{tikzcd}$$

The way we think of this transpose is like the axiom of separation, and comprehension of sets by predicates.

What we're going to do is show that we have exponentials and colimits anyway; the power object is really a special case of the exponential ($PA = \Omega^A$) but it's enough. This gives the axiomatization a much more ZFCish flavor; instead of demanding functions, we just want a power set.

This is also a lot simpler than saying our universe has to look like a bunch of sheaves, as the definition of Grothendieck Topoi does.

Note that we can uniquely extend P to a functor $\mathcal{E}^{\mathrm{op}} \to \mathcal{E}$ by for any $h : B \to C$ saying $Ph := \overline{\in_C (h \times PC)}$ so that

$$\begin{tikzcd}
B \times PC \arrow[r, "h \times PC"] \arrow[d, "B \times Ph"'] \arrow[dr, phantom, "DN"] & C \times PC \arrow[d, "\in_C"] \\
B \times PB \arrow[r, "\in_B"'] & \Omega
\end{tikzcd}$$

commutes.

1.2 Exponentials

We start by defining a few random arrows. Here's our old friend, the diagonal:

$$\Delta_B := (1_B, 1_B)$$

It's monic, so it has a characteristic arrow δ_B:

$$\begin{tikzcd}
B \arrow[rr, "!"] \arrow[d, tail, "\Delta_B"'] \arrow[drr, phantom, "\delta\circ"] \arrow[dr, phantom, very near start, "\lrcorner"] & & 1 \arrow[d, "\mathrm{true}"] \\
B \times B \arrow[rr, "\delta_B := \mathrm{char}\,\Delta_B"'] & & \Omega
\end{tikzcd}$$

$$\begin{tikzcd}
B \arrow[r, "\{\cdot\}_X := \overline{\delta_B}"'] & PB
\end{tikzcd}$$

We can also think that $\delta_B \colon B \times B \to \Omega$ is the predicate $b_1 = b_2$. Then the transpose $\{\cdot\}_B b_1 := \overline{\delta_B} b_1$ is like the set $\{b_2 : B \,|\, b_1 = b_2\}$, that is, $\{b_1\}$. By definition of transpose we have that

$$\begin{tikzcd}
B \times B \arrow[r, "\delta_B"] \arrow[d, "B \times \{\cdot\}_B = B \times \overline{\delta_B}"'] \arrow[dr, phantom, "\{\cdot\}\circ"] & \Omega \arrow[d, equal] \\
B \times PB \arrow[r, "\in_B"'] & \Omega
\end{tikzcd}$$

so in fact for $b, b' \colon X \to B$, we have if $\{\cdot\}_B b = \{\cdot\}_B b'$, then

$$\begin{tikzcd}
X \arrow[r, "b"] \arrow[d, "{(b,1)}"'] \arrow[dr, phantom, "obv"] & B \arrow[r, "!"] \arrow[d, "\Delta_B"] \arrow[dr, phantom, "\delta\circ"] & 1 \arrow[d, "\mathrm{true}"] \\
B \times X \arrow[r, "B \times b"'] \arrow[dd, equal] & B \times B \arrow[r, "\delta_B"'] \arrow[d, "B \times \{\cdot\}_B"] \arrow[dr, phantom, "\{\cdot\}\circ"] & \Omega \arrow[d, equal] \\
 & \mathit{assume}\ B \times PB \arrow[r, "\in_B"'] & \Omega \arrow[d, equal] \\
B \times X \arrow[r, "B \times b'"'] & B \times B \arrow[r, "\delta_B"'] \arrow[u, "B \times \{\cdot\}_B"'] \arrow[ur, phantom, "\{\cdot\}\circ"] & \Omega \\
X \arrow[r, "b'"'] \arrow[u, "{(b',1)}"] \arrow[ur, phantom, "obv"] & B \arrow[r, "!"'] \arrow[u, "\Delta_B"'] \arrow[ur, phantom, "\delta\circ"] & 1 \arrow[u, "\mathrm{true}"']
\end{tikzcd}$$

commutes. We can then check that $!b'$ and $(b',1)B \times b$ satisfy the top right pullback, giving a unique mediating map h, which then has that h and $(b',1)$ satisfy the top left pullback, implying that $(b',1)$ uniquely factors through $(b,1)$. But we can reverse the whole argument and see that $(b,1)$ uniquely factors through $(b',1)$, hence $b = b'$. So $\{\cdot\}_B$ is monic. Hence it, too, has a characteristic function we call σ_B, i.e.

$$\begin{tikzcd}
B \arrow[r, "!"] \arrow[d, tail, "\{\cdot\}_B"'] \arrow[dr, phantom, "\sigma\circ"] & 1 \arrow[d, "\mathrm{true}"] \\
PB \arrow[r, "\sigma_B"'] & \Omega
\end{tikzcd}$$

We have $\in_{C\times B}\colon C\times B\times P(C\times B)\to\Omega$. Define v as $\overline{\in_{C\times B}} : B\times P(C\times B)\to PC$.

$$\begin{tikzcd}
C\times B\times P(C\times B) \arrow[r, "\in_{C\times B}"] \arrow[d, "C\times v"'] \arrow[dr, phantom, "v\circ"] & \Omega \arrow[d, equal] \\
C\times PC \arrow[r, "\in_C"'] & \Omega
\end{tikzcd}$$

That is, $v(b,r) := \{c : C \,|\, (c,b)\in r\}$. Define $u(r) := \overline{\sigma_C v} = \overline{\sigma_C\overline{\in_{C\times B}}}$.

$$\begin{tikzcd}
B\times P(C\times B) \arrow[r, "v"] \arrow[d, "B\times u"'] \arrow[dr, phantom, "u\circ"] & PC \arrow[d, "\sigma_C"] \\
B\times PB \arrow[r, "\in_B"'] & \Omega
\end{tikzcd}$$

That is, $u(b) := \{b : B \,|\, \sigma_C(v(b,r))\}$. Now we pull u pack against the name of true.

$$\begin{tikzcd}
C^B \arrow[r, "!"] \arrow[d, "\pi"'] \arrow[dr, phantom, "exp\circ"] & 1 \arrow[d, "\ulcorner \mathrm{true}_B \urcorner"] \\
P(C\times B) \arrow[r, "u"'] & PB
\end{tikzcd}
\qquad
\begin{tikzcd}
B \arrow[r, "!"] \arrow[d, "\mathrm{true}_B"'] & 1 \arrow[dl, "\mathrm{true}"] \\
\Omega &
\end{tikzcd}$$

Algebraically, we have $C^B := \mathrm{dom}(\ulcorner\mathrm{true}\urcorner^*(\overline{\sigma_C\overline{\in_{C\times B}}}))$.

$$\begin{tikzcd}
B\times C^B \arrow[r, "B\times\pi"] \arrow[d, "B\times !"'] \arrow[rrr, dotted, bend left=40, "e", "e\circ"'] \arrow[dr, phantom, "B\times exp\circ"] & B\times P(C\times B) \arrow[r, "v"] \arrow[d, "B\times u"] \arrow[dr, phantom, "u\circ"] & PC \arrow[d, "\sigma_C"] \arrow[dr, phantom, "\sigma\circ"] & C \arrow[l, "\{\cdot\}_C"'] \arrow[d, "!"] \\
B\times 1 \arrow[r, "B\times\ulcorner\mathrm{true}_B\urcorner"'] \arrow[d, equal] \arrow[drr, phantom, "\text{(definition of transpose)}"] & B\times PB \arrow[r, "\in_B"'] & \Omega & 1 \arrow[l, "\mathrm{true}"] \\
B\times 1 \arrow[rr, equal] & & B\times 1 \arrow[u, "\mathrm{true}_B"] \arrow[r, "\pi_1"'] & B \arrow[u, "!"']
\end{tikzcd}$$

We get by chasing the diagram (because $\sigma\circ$ is a pbk) a unique $e\colon B\times C^B \to C$ that makes ($e\circ$) commute. Is this the evaluation map? We would want that for every $f\colon B\times A \to C$ there is a unique $g\colon A \to C^B$ with $f = e(B\times g)$. Now *if* this equation holds, then we have that the diagram

$$\begin{tikzcd}
B\times A \arrow[r, "f"] \arrow[dr, "B+g"'] & C \arrow[r, "\{\cdot\}_C"] \arrow[dr, phantom, "e\circ"] & PC \\
& B\times C^B \arrow[u, "e"] \arrow[r, "B\times\pi"'] & B\times P(C\times B) \arrow[u, "v"']
\end{tikzcd}$$

commutes. We can hit this with $C \times —$ and get

$$\begin{tikzcd}
 & & & \Omega \\
C \times B \times A \arrow[r, "C \times f"] \arrow[dr, "C \times B \times g"'] & C \times C \arrow[r, "C \times \{\cdot\}_C"] \arrow[urr, bend left, "\delta_C"] & C \times PC \arrow[ur, "\in_C"] & \\
 & C \times B \times C^B \arrow[u, "C \times e"] \arrow[r, "C \times B \times \pi"'] \arrow[ur, phantom, "C \times e\circ"] & C \times B \times P(C \times B) \arrow[u, "C \times v"] \arrow[uur, "\in_{C \times B}"'] \arrow[ur, phantom, "v\circ"] &
\end{tikzcd}$$

Now we see that πg is the transpose of $\delta_C(C \times f)$, and π is monic, so g is uniquely determined by f. Now to find g, run the argument in reverse; we have f, so find the transpose h of $\delta_C(C \times f) = \in_{C \times B} (C \times B \times h)$. Transposing both sides of this gives $\{\cdot\}_C f = v(B \times h)$. Compose on the left with σ_C to get $\mathrm{true}_C f = \sigma_C v(B \times h)$. It's just a diagram chase to see $\mathrm{true}_C f = \mathrm{true}_B\, \mathrm{fst}_{B \times A}$.

$$\begin{tikzcd}
 & B \arrow[dr, "!"] & \\
B \times A \arrow[ur, "\mathrm{fst}_{B \times A}"] \arrow[r, "f"] \arrow[d, "B \times h"'] & C \arrow[r, "!"] \arrow[d, "\{\cdot\}_C"] \arrow[dr, phantom, "\sigma\circ"] & 1 \arrow[d, "\mathrm{true}"] \\
B \times P(B \times C) \arrow[r, "v"'] & PC \arrow[r, "\sigma_C"'] & \Omega
\end{tikzcd}$$

Now transpose both sides to get $\ulcorner \mathrm{true}_B \urcorner \circ !_A = uh$. But this is the pullback condition on $exp\circ$.

$$\begin{tikzcd}
 & C^B \arrow[r, "!"] \arrow[d, "\pi"] \arrow[dr, phantom, "exp\circ"] & 1 \arrow[d, "\ulcorner \mathrm{true}_B \urcorner"] \\
A \arrow[r, "h"'] \arrow[ur, dashed, "g"] & P(C \times B) \arrow[r, "u"'] & PB
\end{tikzcd}$$

It is now "easily checked" that g is the transpose of f.

1.3 Direct Image

Given a monic $k : B' \to B$ we want to construct $\exists_k : PB' \to PB$, the 'direct image under k'.

$$\begin{tikzcd}
U_B \arrow[r, "!"] \arrow[d, dashed, "u_{B'}"'] \arrow[dr, phantom, "u\circ"] & 1 \arrow[r, equal] \arrow[d, "\mathrm{true}"] & 1 \arrow[dd, "\mathrm{true}"] \\
B' \times PB' \arrow[r, "\in_{B'}"'] \arrow[d, "k \times PB'"'] & \Omega & \\
B \times PB' \arrow[rr, "e_k"'] & & \Omega
\end{tikzcd}$$

$$PB' \xrightarrow{\exists_k} PB$$

We bull back true across $\in_{B'}$ to get a monic $u_{B'}$, compose it with the monic $k \times PB'$, and find its characteristic map e_k. The transpose of this is $\exists_k$. Think of this as (for some $T \in B'$) saying $\exists_k(T) = \{b \in B \,|\, (b, T) \in (k \times PB')(u_{B'})\}$. i.e. $\exists_k(T) = \{b \in B \mid \exists b' : k(b') = b \wedge b' \in T\}$.

1.4 Beck-Chevalley

Proposition 1.1 *If*

$$\begin{tikzcd}
C' \arrow[r, "g'"] \arrow[d, tail, "m"'] \arrow[dr, phantom, "\lrcorner", very near start] & B' \arrow[d, tail, "k"] \\
C \arrow[r, "g"'] & B
\end{tikzcd}$$

then

$$\begin{tikzcd}
PC' \arrow[d, "\exists_m"'] & PB' \arrow[l, "Pg'"'] \arrow[d, "\exists_k"] \\
PC & PB \arrow[l, "Pg"]
\end{tikzcd}$$

Proof It suffices to show that $e_k(g \times PB') = e_m(C \times Pg')$, or prove the equality of the subobjects characterized by these arrows.

$$\begin{tikzcd}
? \arrow[r, dashed] \arrow[d, dashed] & U_{B'} \arrow[r, "!"] \arrow[d, "u_{B'}"] \arrow[dr, phantom, "\lrcorner", very near start] & 1 \arrow[dd, "\mathrm{true}"] & \\
C' \times PB' \arrow[r, "g' \times PB'"] \arrow[d, "m \times PB'"'] \arrow[dr, phantom, "\lrcorner", very near start] & B' \times PB' \arrow[d, "k \times PB'"] & & \mathfrak{P}_1 \\
C \times PB' \arrow[r, "g \times PB'"'] & B \times PB' \arrow[r, "e_k"'] & \Omega &
\end{tikzcd}$$

$$\begin{tikzcd}
? \arrow[r, dashed] \arrow[d, dashed] & U_{C'} \arrow[r, "!"] \arrow[d, "u_{C'}"] \arrow[dr, phantom, "\lrcorner", very near start] & 1 \arrow[dd, "\mathrm{true}"] & \\
C' \times PB' \arrow[r, "C' \times Pg'"] \arrow[d, "m \times PB'"'] \arrow[dr, phantom, "\lrcorner", very near start] & C' \times PC' \arrow[d, "m \times PC'"] & & \mathfrak{P}_2 \\
C \times PB' \arrow[r, "C \times Pg'"'] & C \times PC' \arrow[r, "e_m"'] & \Omega &
\end{tikzcd}$$

Now consider diagram DN instantiated as follows:

$$\begin{tikzcd}
C' \times PB' \arrow[r, "g' \times PB'"] \arrow[d, "C' \times Pg'"'] \arrow[dr, phantom, "DN_{g'}"] & B' \times PB' \arrow[d, "\in_{B'}"] \\
C' \times PC' \arrow[r, "\in_{C'}"'] & \Omega
\end{tikzcd}$$

Specifically consider pulling back true : $1 \to \Omega$ across both paths in it; $DN_{g'}$ commutes, so pulling back across both complete paths gives the same result. But pulling back true across the top of DN is first pulling back across $\in_{B'}$ and then across $g' \times PB'$. But $\in_C^*$ true $= u'_C$ by $u\circ$, and the pullback of u'_C across $g' \times PB'$ is exactly the top left corner of $\mathfrak{P}_1$. Similarly the pullback of true across the bottom side of $DN_{g'}$ is the top left corner of $\mathfrak{P}_2$. So as subobjects these are equal, and since characteristic arrows are unique, we know $e_m(C \times Pg') = e_k(g \times PB')$. ■

Corollary 1.2 *The composite*

$$PB' \xrightarrow{\exists_k} PB \xrightarrow{Pk} PB'$$

is the identity.

Proof Hit

$$\begin{tikzcd}
B' \arrow[r, "\mathrm{id}"] \arrow[d, "\mathrm{id}"'] \arrow[dr, phantom, "\lrcorner", very near start] & B' \arrow[d, tail, "k"] \\
B' \arrow[r, tail, "k"'] & B
\end{tikzcd}$$

with Beck-Chevalley. $P\mathrm{id} = \mathrm{id}$ and $\exists_{\mathrm{id}} = \mathrm{id}$. ■

2 Monads and Beck's Theorem

2.1 Monads

A monad on $\mathbf{C}$ is a functor $T : \mathbf{C} \to \mathbf{C}$, a natural transformation $\mu : T^2 \to T$ and a natural transformation $\eta : 1_{\mathbf{C}} \to T$ such that the diagrams

$$\begin{tikzcd}
T^3 \arrow[r, "T\mu"] \arrow[d, "\mu_T"'] & T^2 \arrow[d, "\mu"] \\
T^2 \arrow[r, "\mu"'] & T
\end{tikzcd}
\qquad
\begin{tikzcd}
 & T^2 \arrow[d, "\mu"] & \\
T \arrow[ur, "\eta T"] \arrow[r, equal] & T & T \arrow[l, equal] \arrow[ul, "T\eta"']
\end{tikzcd}$$

commute. Each adjunction

$$\langle F \dashv G, \eta, \varepsilon \rangle \colon \mathbf{C} \to \mathbf{A}$$

induces an monad $\langle GF, G\varepsilon_F, \eta \rangle$.

$$\begin{tikzcd}[column sep=large]
\mathbf{A} & \mathbf{A} \arrow[l, bend right=60, "FG"'] \arrow[l, "1_{\mathbf{A}}" description] \arrow[l, bend left=60, "1_{\mathbf{A}}"] \arrow[l, phantom, shift right=3, "\Downarrow \varepsilon"] \arrow[l, phantom, shift left=3, "\Downarrow 1_{1_{\mathbf{A}}}"] & \mathbf{A} \arrow[l, bend right=60, "FG"'] \arrow[l, "FG" description] \arrow[l, bend left=60, "1_{\mathbf{A}}"] \arrow[l, phantom, shift right=3, "\Downarrow 1_{FG}"] \arrow[l, phantom, shift left=3, "\Downarrow \varepsilon"] & \mathbf{A} \arrow[l, bend right=60, "FG"'] \arrow[l, "1_{\mathbf{A}}" description] \arrow[l, bend left=60, "1_{\mathbf{A}}"] \arrow[l, phantom, shift right=3, "\Downarrow \varepsilon"] \arrow[l, phantom, shift left=3, "\Downarrow 1_{1_{\mathbf{A}}}"]
\end{tikzcd}$$

By staring at the above diagram, we can see that (left) $\varepsilon\varepsilon = \varepsilon\varepsilon_{FG}$ and (right) $\varepsilon\varepsilon = \varepsilon FG\varepsilon$. Left-composing with G and right-composing with F gives

$$G\varepsilon_F G\varepsilon_{FGF} = G\varepsilon_F GFG\varepsilon_F$$

, the associative law for the monad. The unit laws are $G\varepsilon_F\eta_{GF}$ and $G\varepsilon_F GF\eta$, which follow from the triangle identities $G\varepsilon\eta_G = \mathrm{id}$ and $\varepsilon_F F\eta = \mathrm{id}$.

2.2 Algebras

A T-algebra for a monad T generalizes the notion of a monoid action; A T-algebra is a pair (C, h) such that $h : TC \to C$ and the following diagrams commute:

$$
\begin{tikzcd}
T^2C \arrow[r, "Th"] \arrow[d, "\mu_C"'] & TC \arrow[d, "h"] \\
TC \arrow[r, "h"'] & C
\end{tikzcd}
\qquad
\begin{tikzcd}
C \arrow[r, "\eta_C"] \arrow[d, equal] \arrow[dr, phantom, "\bullet", near start] & TC \arrow[dl, "h"] \\
C &
\end{tikzcd}
$$

These correspond to the associative law $(gh) \cdot x = g \cdot (h \cdot x)$ and the unit law $e \cdot x = x$ of a monoid action. A T-algebra morphism $(C, h) \to (C', h')$ is an arrow f such that

$$
\begin{tikzcd}
TC \arrow[r, "Tf"] \arrow[d, "h"'] & TC' \arrow[d, "h'"] \\
C \arrow[r, "f"'] & C'
\end{tikzcd}
$$

commutes. It is easy to check that T-algebras and morphisms thereof form a category, called $\mathbf{C}^T$. For any monad T, there is an adjunction $\langle F^T \dashv G^T, \eta^T, \mu^T\rangle : \mathbf{C} \to \mathbf{C}^T$. G^T is the evident "forgetful" functor which forgets the structure map. F^T is the "free T-algebra" functor, defined by $F^T(C) := (TC, \mu_C)$ and $F^T(f\colon C \to C') := Tf\colon (TC, \mu_C) \to (TC', \mu_{C'})$. That $F^T(C)$ is a T-algebra follows from the fact that T is a monad; T's associativity diagram is exactly $F^T(C)$'s, and $F^T(C)$'s unit law follows from one of T's.

The diagram to check to show that $F^T(f)$ is a morphism of T-algebras is

$$\begin{tikzcd}
T^2C \arrow[r, "T^2f"] \arrow[d, "\mu_C"'] & T^2C' \arrow[d, "\mu_{C'}"] \\
TC \arrow[r, "Tf"'] & TC'
\end{tikzcd}$$

which commutes because μ is natural.

Now suppose we have an adjunction $\langle F \dashv G, \eta, \mu\rangle : \mathbf{C} \to \mathbf{A}$. The picture of the world then looks like

$$\begin{tikzcd}
\mathbf{A} \arrow[d, shift left, "G"] & \mathbf{C}^T \arrow[d, shift left, "G^T"] \\
\mathbf{C} \arrow[u, shift left, "F"] \arrow[u, phantom, "\dashv"] \arrow[r, equal] & \mathbf{C} \arrow[u, shift left, "F^T"] \arrow[u, phantom, "\dashv"]
\end{tikzcd}$$

what we want to do is fill in the top, so things look like

$$\begin{tikzcd}
\mathbf{A} \arrow[r, dashed, "K"] \arrow[d, shift left, "G"] & \mathbf{C}^T \arrow[d, shift left, "G^T"] \\
\mathbf{C} \arrow[u, shift left, "F"] \arrow[u, phantom, "\dashv"] \arrow[r, equal] & \mathbf{C} \arrow[u, shift left, "F^T"] \arrow[u, phantom, "\dashv"]
\end{tikzcd}$$

so that $KF = F^T$ and $G^TK = G$. The latter condition implies that the underlying object of the T-algebra KA has to be GA. So $KA = (GA, f(A))$, for some f. $G^TK = G$ also says that K must act the same as G on arrows, since G^T is the inclusion of $\mathbf{C}^T$-arrows into $\mathbf{C}$. $KF = F^T$ says that $K(FC) = F^T(C) = (GFC, G\varepsilon_{FC})$, so the logical choice is $KA = (GA, G\varepsilon_A)$. Indeed this choice (and so the whole 'comparison functor') is unique if you think about adjunctions for a while, and this fact is called the 'comparison theorem' by MacLane.

Now in most algebraic cases, this K is an equivalence, in which case we call G *monadic*. The nice thing about monadic functors is that they create limits. In fact all we need to show to see this is that the functor $G^T : \mathbf{C}^T \to \mathbf{C}$ creates limits, since if G is monadic, $\mathbf{A} \sim \mathbf{C}^T$, and so $G : \mathbf{A} \to \mathbf{C}$ creates limits just in case G^T does. So,

Proposition 2.1 *G^T creates limits.*

Proof Let a diagram $D : J \to \mathbf{C}^T$ be given; Say $D(j) = (C_j, h_j)$ for each object $j \in J$. Assume there is a limiting cone $k : \Delta(L) \to G^T \circ D$ where L is an object of $\mathbf{C}$ and Δ is the diagonal functor $\mathbf{C} \to \mathbf{C}^J$. So for each object $j \in J$ there is an arrow $k_j\colon L \to C_j$, and for each arrow $f\colon j \to j'$ in J we have that

$$\begin{tikzcd} L \arrow[r, "k_j"] \arrow[d, equal] & C_j \arrow[d, "D(f)"] \\ L \arrow[r, "k_{j'}"'] & C_{j'} \end{tikzcd}$$

commutes. By the definition of creation of limits and G^T, what we need to do is find a structure map ℓ so that (L, ℓ) is a T-algebra and all the k_j are T-algebra morphisms. That is, we want the diagram

$$\begin{tikzcd} TL \arrow[r, "Tk_j"] \arrow[d, dashed, "\ell"'] & TC_j \arrow[d, "h_j"] \\ L \arrow[r, "k_j"'] & C_j \end{tikzcd}$$

to commute for all j. Life would be wonderful if $(h_j \circ Tk_j)$ were a cone over the diagram $G^T D$.

$$\begin{tikzcd} TL \arrow[r, "Tk_j"] \arrow[d, equal] & TC_j \arrow[r, "h_j"] \arrow[d, "TD(f)"] & C_j \arrow[d, "D(f)"] \\ TL \arrow[r, "Tk_{j'}"'] & TC_{j'} \arrow[r, "h_{j'}"'] & C_{j'} \end{tikzcd}$$

The left square commutes because it's just the fact that k is a cone hit with T. The right square commutes because $D(f)$ has got to be a T-algebra morphism. So h_jTk_j is a cone, and so there is a unique ℓ making L into a T-algebra, as required. It can easily be checked that (L,ℓ) is limiting: For suppose $h\colon \Delta(M,m) \to D$ for (M,m) a T-algebra is a natural transformation, i.e. a cone over D. Then $G^T h\colon \Delta(M) \to G^T \circ D$ is a cone over $G^T \circ D$, hence there is a unique arrow $z\colon \Delta(M) \to L$ such that

$$\begin{tikzcd}
M \arrow[d, dotted, "z"'] \arrow[dr, "h_{(M,m)}"] & \\
L \arrow[r, "k_j"'] & C_j
\end{tikzcd}$$

commutes. But then we can see from the diagram

$$\begin{tikzcd}
TM \arrow[r, "m"] \arrow[d, dotted, "Tz"'] \arrow[ddr] & M \arrow[d, "z"] \arrow[ddr, "h_{(M,m)}"] & \\
TL \arrow[r, "\ell"] \arrow[dr, "Tk_j"'] & L \arrow[dr, "k_j"] & \\
 & TC_j \arrow[r, "h_j"'] & C_j
\end{tikzcd}$$

(the unlabelled arrow $TM \to TC_j$ is $Th_{(M,m)}$) that $k_jzm = k_j\ell Tz$ for all j, so $zm = \ell Tz$ and z is a T-algebra morphism. (any arrows into a limit which are equal after left-composing with all projections out of the limit are equal?)
■

2.3 Beck's Theorem

Now we want some way of guaranteeing the existence of monadic functors. MacLane calls this the Weak Tripleability Theorem.

Proposition 2.2

1. *If* $\mathbf{A}$ *has coequalizers of reflexive pairs, then* K *has a left adjoint,* L*.*

2. *If* G *also preserves these coequalizers, then the unit of this adjunction is an isomorphism* $\mathrm{id}_{\mathbf{C}^T} \cong KL$*.*

3. *If* G *also reflects isomorphisms, then the counit of the adjunction is an isomorphism* $LK \cong \mathrm{id}_{\mathbf{C}}$*.*

$$\begin{tikzcd}
\mathbf{A} \arrow[r, shift left=2, "K"] \arrow[r, phantom, "\top"] \arrow[d, shift left=2, "G"] \arrow[d, phantom, "\dashv"] & \mathbf{C}^T \arrow[l, shift left=2, "L"] \arrow[d, shift left=2, "G^T"] \arrow[d, phantom, "\dashv"] \\
\mathbf{C} \arrow[u, shift left=2, "F"] \arrow[r, equal] & \mathbf{C} \arrow[u, shift left=2, "F^T"]
\end{tikzcd}$$

Proof

1. Let $L(C,h)$ be the coequalizer

$$FGFC \underset{\varepsilon_{FC}}{\overset{Fh}{\rightrightarrows}} FC \overset{e}{\dashrightarrow} L(C,h) \qquad (*)$$

This pair is reflexive because $FhF\eta_C = F(h\eta_C) = \mathrm{id}$ by the identity law for a T-algebra and $\varepsilon_{FC}F\eta_C$ by the triangle identity. Now suppose we have some arrow $f\colon (C,h) \to KA \in \mathbf{C}^T$ for some $A \in \mathbf{A}$. This means we have a diagram

$$\begin{tikzcd}
GFC \arrow[r, "GFf"] \arrow[d, "h"'] \arrow[dr, phantom, "\flat"] & GFGA \arrow[d, "G\varepsilon_A"] \\
C \arrow[r, "f"'] & GA
\end{tikzcd}$$

Since $F \dashv G$, we know there is a unique arrow $\bar{f}\colon FC \to A$ such that

$$\begin{tikzcd}
GFC \arrow[r, "G\bar{f}"] & GA \\
C \arrow[u, "\eta_C"] \arrow[ur, "f"'] \arrow[r, phantom, "\natural"] &
\end{tikzcd}$$

commutes. The dual adjunction diagram

$$\begin{tikzcd}
FC \arrow[r, "Ff"] \arrow[dr, "\bar{f}"'] & FGA \arrow[d, "\varepsilon_A"] \arrow[dl, phantom, "\sharp"] \\
& A
\end{tikzcd}$$

also commutes. combining diagrams we get

$$\begin{tikzcd}
FGFC \arrow[rrr, equal, bend left] \arrow[r, "FGFf"] \arrow[d, "Fh"'] \arrow[dr, phantom, "F\flat"] & FGFGA \arrow[r, equal] \arrow[d, "FG\varepsilon_A"] & FGFGA \arrow[d, "\varepsilon_{FGA}"] & FGFC \arrow[l, "FGFf"'] \arrow[d, "\varepsilon_{FC}"] \arrow[dl, phantom, "\varepsilon\text{ nat.}"] \\
FC \arrow[r, "Ff"] \arrow[dr, "\bar{f}"'] & FGA \arrow[d, "\varepsilon_A"] \arrow[dl, phantom, "\sharp"] & FGA \arrow[d, "\varepsilon_A"] & FC \arrow[l, "Ff"'] \arrow[dl, "\bar{f}"] \arrow[dl, phantom, "\sharp"] \\
& A \arrow[r, equal] & A &
\end{tikzcd}$$

But remember that $L(C, h)$ is a coequalizer, so there is a unique k such that

$$\begin{tikzcd} & & & A \\ FGFC \arrow[r, shift left, "Fh"] \arrow[r, shift right, "\varepsilon_{FC}"'] & FC \arrow[urr, "\bar{f}"] \arrow[rr, "e"'] & \star & L(C,h) \arrow[u, dashed, "k"'] \end{tikzcd}$$

Notice that

$$\begin{tikzcd} GFGFC \arrow[r, shift left, "G\varepsilon_{FC}"] \arrow[r, shift right, "GFh"'] & GFC \arrow[r, "h"] & C \end{tikzcd} \qquad (\clubsuit)$$

is a split coequalizer; that is,

$$\begin{tikzcd} GFC \arrow[rr, equal] \arrow[dr, "\eta_{GFC}"] \arrow[ddd, "h"'] & & GFC \arrow[ddd, "h"] \\ & GFGFC \arrow[ur, "G\varepsilon_{FC}"] \arrow[d, "GFh"] & \\ & GFC \arrow[dr, "h"] & \\ C \arrow[ur, "\eta_C"] \arrow[rr, equal] & & C \end{tikzcd}$$

commutes. But if we hit the bottom row of $(*)$ with G, we see (since Ge coequalizes $\clubsuit$'s parallel pair) there has to be a unique $u : C \to GL(C,h)$

such that $uh = Ge$ like so:

$$\begin{tikzcd}
GFGFC \arrow[r, shift left, "G\varepsilon_{FC}"] \arrow[r, shift right, "GFh"'] \arrow[d, equal] & GFC \arrow[r, "Ge"] \arrow[d, equal] \arrow[dr, phantom, "\ddagger"] & GL(C,h) \\
GFGFC \arrow[r, shift left, "G\varepsilon_{FC}"] \arrow[r, shift right, "GFh"'] & GFC \arrow[r, "h"'] & C \arrow[u, dashed, "u"']
\end{tikzcd}$$

But if we hit both sides of this equation with η_C on the right we see $u = u(h\eta_C) = Ge\eta_C$. That means u fits into the diagram

$$\begin{tikzcd}
GFC \arrow[r, "Ge"] & GL(C,h) \\
C \arrow[u, "\eta_C"] \arrow[ur, "u"'] \arrow[ur, phantom, shift left=3, "\S"]
\end{tikzcd}$$

so u is really the adjoint transpose $\tilde{e}$ of e! We also then get that

$$\begin{tikzcd}
FGL(C,h) \arrow[d, "\varepsilon_{L(C,h)}"'] & FC) \arrow[l, "F\tilde{e}"'] \arrow[dl, "e"] \arrow[dl, phantom, shift right=3, "\dagger"] \\
L(C,h)
\end{tikzcd}$$

commutes. We can then conclude that

$$\begin{tikzcd}
GFGL(C,h) \arrow[d, "G\varepsilon_{L(C,h)}"'] \arrow[dr, phantom, shift left=2, "G\dagger"] & TC \arrow[l, "T\tilde{e}"'] \arrow[dl, "Ge"] \arrow[d, "h"] \arrow[dl, phantom, shift right=4, "\ddagger"] \\
GL(C,h) & C \arrow[l, "\tilde{e}"]
\end{tikzcd}$$

That is, $\tilde{e}$ is a T-algebra morphism from (C, h) to $KL(C, h)$. So what we want to conclude is that there is a unique k such that

$$\begin{tikzcd}
KL(C,h) \arrow[r, "Kk"] & KA \\
(C,h) \arrow[u, "\tilde{e}"] \arrow[ur, "f"'] &
\end{tikzcd}$$

commutes; then $L \dashv K$ with unit $\tilde{e}$. but recall that the action of K on arrows is G's, and composition of arrows in $\mathbf{C}^T$ is just composition of arrows in $\mathbf{C}$, so we need

$$\begin{tikzcd}
GL(C,h) \arrow[r, "Gk"] & GA \\
C \arrow[u, "\tilde{e}"] \arrow[ur, "f"'] &
\end{tikzcd}$$

to commute. But we have for the k that the coequalizer produced that

$$\begin{tikzcd}
GL(C,h) \arrow[rr, "Gk"] & & GA \arrow[dd, equal] \\
\arrow[r, phantom, "\S"] & GFC \arrow[ul, "Ge"] \arrow[dr, "G\bar{f}"] \arrow[ur, phantom, "G\star"] & \\
C \arrow[uu, "\tilde{e}"] \arrow[ur, "\eta_C"] \arrow[rr, "f"'] \arrow[urr, phantom, "\natural"] & & GA
\end{tikzcd}$$

We conclude $L \dashv K$.

2. Suppose G preserves coequalizers. Then both rows of the following are

coequalizers, hence $u = \tilde{e}$ is an isomorphism.

$$\begin{tikzcd}
GFGFC \arrow[r, shift left, "G\varepsilon_{FC}"] \arrow[r, shift right, "GFh"'] \arrow[d, equal] & GFC \arrow[r, "h"] \arrow[d, equal] & C \arrow[d, dashed, "u"] \\
GFGFC \arrow[r, shift left, "G\varepsilon_{FC}"] \arrow[r, shift right, "GFh"'] & GFC \arrow[r, "Ge"'] & GL(C,h)
\end{tikzcd}$$

3. If we take (C,h) to be $KA = (GA, G\varepsilon_A)$, then for $(*)$ we get a diagram (remembering the interchange law result)

$$\begin{tikzcd}
FGFGA \arrow[r, shift left, "FG\varepsilon_A"] \arrow[r, shift right, "\varepsilon_{FGA}"'] & FGA \arrow[r, two heads, "e_{KA}"] \arrow[dr, "\varepsilon_A"'] & LKA \arrow[d, dashed, "\kappa_A"] \\
 & & A
\end{tikzcd} \qquad (*_{KA})$$

κ_A ends up being the counit of the adjunction. But wait, the following is a split coequalizer:

$$GF\triangle$$

$$\begin{tikzcd}
GFGA \arrow[r, "GF\eta_{GA}"] \arrow[d, "G\varepsilon_A"'] \arrow[dr, phantom, "(G\varepsilon)\ \text{nat}"] & GFGFGA \arrow[r, "GFG\varepsilon_A"] \arrow[d, "G\varepsilon_{FGA}"] \arrow[dr, phantom, "G\ \text{interch}"] & GFGA \arrow[d, "G\varepsilon_A"] \\
GA \arrow[r, "\eta_{GA}"'] & GFGA \arrow[r, "G\varepsilon_A"'] & GA
\end{tikzcd}$$

$$\triangle$$

so we stare at

$$\begin{tikzcd}
GFGFGA \arrow[r, shift left, "GFG\varepsilon_A"] \arrow[r, shift right, "G\varepsilon_{FGA}"'] \arrow[d, equal] & GFGA \arrow[r, two heads, "Ge_{KA}"] \arrow[d, equal] \arrow[dr, "G\varepsilon_A" description] & GLKA \arrow[d, dashed, "G\kappa_A"] \\
GFGFGA \arrow[r, shift left, "GFG\varepsilon_A"] \arrow[r, shift right, "G\varepsilon_{FGA}"'] & GFGA \arrow[r, two heads, "G\varepsilon_A"'] & GA
\end{tikzcd} \qquad (G*_{KA})$$

(remembering that G is supposed to preserve equalizers as well) and realize that $G\kappa_A$ is an isomorphism. If G reflects isomorphisms, then κ_A is. ∎

3 Colimits

3.1 Construction of Colimits

We get colimits in an arbitrary topos out of properties of the powerset functor.

Proposition 3.1 *Any topos is balanced. That is, every arrow which is mono and epi is an isomorphism.*

Proof Every monic arrow $m : S \to B$ is the equalizer of $\mathrm{char}\, m$ and true_S. But an epi regular monic is always an isomorphism; suppose e monic and the equalizer of f, h. If e is epi, then $fe = ge$ implies $f = g$. But the equalizer of f with itself is always an isomorphism. ∎

Proposition 3.2 *The powerset functor $P : \mathcal{E}^{\mathrm{op}} \to \mathcal{E}$ is faithful.*

Proof

$$\begin{tikzcd}
B \arrow[r, "h"] \arrow[d, "\langle 1, h\rangle"'] \arrow[dr, phantom, "obv"] & A \arrow[r, "!"] \arrow[d, "\Delta_A"] \arrow[dr, phantom, "\delta\circ"] & 1 \arrow[d, "\mathrm{true}"] \\
B \times A \arrow[r, "h \times A"'] & A \times A \arrow[r, "\delta_A"'] & \Omega
\end{tikzcd}$$

the thing on the bottom is the characteristic arrow of $\langle 1, h\rangle$, and the following diagram commutes:

$$\begin{tikzcd}
B \times A \arrow[r, "B \times \{\cdot\}_A"] \arrow[d, "h \times A"'] \arrow[dr, phantom, "obv"] & B \times PA \arrow[r, "B \times Ph"] \arrow[d, "h \times PA"] \arrow[dr, phantom, "DN"] & B \times PB \arrow[d, "\in_B"] \\
A \times A \arrow[r, "A \times \{\cdot\}_A"'] \arrow[rr, bend right=60, "\delta_A"'] \arrow[rr, phantom, bend right=30, "\{\cdot\}\circ"] & A \times PA \arrow[r, "\in_A"'] & \Omega
\end{tikzcd}$$

So $Ph\{\cdot\}_A$ is the transpose of char $\langle 1, h\rangle$. This means that if $Ph = Pk$ then of course $Ph\{\cdot\}_A = Pk\{\cdot\}_A$ and so $h = k$ using all the uniqueness of the topos definition. That is, P is faithful. ■

Proposition 3.3 *P is monadic.*

Proof P's left adjoint is 'itself'.

$$\mathcal{E}(A, PB) \cong \mathcal{E}(B \times A, \Omega) \cong \mathcal{E}(A \times B)\Omega \cong \mathcal{E}(B, PA) \cong \mathcal{E}^{\mathrm{op}}(P^{\mathrm{op}}A, B)$$

Well, $P^{\mathrm{op}} : \mathcal{E} \to \mathcal{E}^{\mathrm{op}}$, actually.

Now we want to use Beck's Theorem. $\mathcal{E}^{\mathrm{op}}$ has all coequalizers, because $\mathcal{E}$ has all finite limits. Because P is faithful, it reflects monics and epis. But topoi are balanced, so P reflects isomorphisms. We just need to show that P preserves enough coequalizers. Consider a coequalizer diagram in $\mathcal{E}^{\mathrm{op}}$:

$$\begin{tikzcd}
C \arrow[r, "g"] & B \arrow[r, shift left, "h"] \arrow[r, shift right, "k"'] & A
\end{tikzcd}$$

We really only need to consider preservation of coequalizers of "coreflexive pairs" so we might as well assume we have $d : A \to B$ with $dh = dk = 1$. We need to show that

$$\begin{tikzcd}
PA \arrow[r, shift left, "Ph"] \arrow[r, shift right, "Pk"'] & PB \arrow[r, "Pg"] & PC
\end{tikzcd}$$

is a coequalizer in $\mathcal{E}$. Since the pair is coreflexive, we know

$$\begin{tikzcd} C \arrow[r, "g"] \arrow[d, "g"'] \arrow[dr, phantom, "\spadesuit"] & B \arrow[d, "h"] \\ B \arrow[r, "k"'] & A \end{tikzcd}$$

is a pullback; if $hf = kf'$, then $dhf = dhf'$ so $f = f'$. That is, $f = f'$ equalizes h, k so we get the right unique map. g is an equalizer, so it's monic, and d means that h, k are sections, so monic. But internal Beck-Chevalley (since $\spadesuit$ is a pullback) says the following is a split coequalizer:

$$\begin{tikzcd} PB \arrow[r, "\exists_h"] \arrow[d, "Pg"'] \arrow[dr, phantom, "BC(\spadesuit)"] & PA \arrow[r, "Ph"] \arrow[d, "Pk"] \arrow[dr, phantom, "P\spadesuit"] & PB \arrow[d, "Pg"] \\ PC \arrow[r, "\exists_g"'] & PB \arrow[r, "Pg"'] & PC \end{tikzcd}$$

So P does preserve coequalizers. Beck's Theorem kicks in and gives us P monadic. ∎

Corollary 3.4 *$\mathcal{E}$ has all finite colimits.*

Proof Let $T = PP^{\mathrm{op}}$ be the monad defined by $P^{\mathrm{op}} \dashv P$.

$$\begin{tikzcd} \mathcal{E}^{\mathrm{op}} \arrow[rr, "K"] \arrow[d, shift left, "P"] & & \mathcal{E}^T \arrow[d, shift left, "G^T"] \\ \mathcal{E} \arrow[u, shift left, "P^{\mathrm{op}}"] \arrow[rr, equal] & & \mathcal{E} \arrow[u, shift left, "F^T"] \end{tikzcd}$$

$\mathcal{E}$ has all limits, so since G^T creates limits we know that $\mathcal{E}^T$ has all limits. But since Beck's theorem guarantees that K is an equivalence, so $\mathcal{E}^{\mathrm{op}}$ has all limits, hence $\mathcal{E}$ has all colimits. ∎

Corollary 3.5 *If a topos has all limits, it has all colimits?*

3.2 Explicit Calculation of the Coproduct

Let two sets A, B be given. We're going to compute the coproduct of them from the previous theorem. The relevant diagram is the functor $H : 2 \to \mathbf{Sets}$ with $H(0) = A$ and $H(1) = B$. Now H^{op} is really the same diagram, only considered as a functor $H^{\mathrm{op}} : 2^{\mathrm{op}} \to \mathbf{Sets}^{\mathrm{op}}$. We hit H^{op} with P and get a diagram $PH^{\mathrm{op}} : 2^{\mathrm{op}} \to \mathbf{Sets}$. Now **Sets**, being a topos, has all limits, so we can take the limit of this diagram, which happens to be $PA \times PB$, with projections $\pi_1 : PA \times PB \to PA$ and $\pi_2 : PA \times PB \to PB$. The first thing we need to do is lift this limit into the category $\mathbf{Sets}^{PP^{\mathrm{op}}}$ of PP^{op}-algebras. This is achieved by staring at the diagram

$$\begin{tikzcd}
P^{\mathrm{op}}PA \arrow[d, "P\varepsilon_A"'] & PP^{\mathrm{op}}(PA \times PB) \arrow[l, "PP^{\mathrm{op}}\pi_1"'] \arrow[r, "PP^{\mathrm{op}}\pi_2"] \arrow[d, dotted, "\ell"'] & PP^{\mathrm{op}}PB \arrow[d, "P\varepsilon_B"] \\
PA & PA \times PB \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & PB
\end{tikzcd}$$

We get a unique arrow ℓ that makes these two commute. But we want to understand exactly what this ℓ is; It's the pairing of the two arrows into PA and PB, so we'll try to understand them. Now the right column is $A \in \mathbf{Sets}^{\mathrm{op}}$ hit with the comparison functor. Remember, we showed that G^T creates limits, but we need P to create them. So we use the fact that K is an equivalence implicitly here.

The thing is, we already have a lot of powersets floating around. Say a *family* in A is an element of $P^{\mathrm{op}}PA$. Think about ε; it's a natural transformation from $P^{\mathrm{op}}P$ to $\mathrm{id}_{\mathbf{Sets}^{\mathrm{op}}}$. That means that ε_A's an arrow in $\mathbf{Sets}^{\mathrm{op}}$ from $P^{\mathrm{op}}PA$ to A. In other words, an arrow from A to $P^{\mathrm{op}}PA$ in **Sets**; but we remember that this has to be the arrow which takes an element to the family of subsets containing it. Say a family F *represents* an element x if $\varepsilon(x) = F$. Every element has a unique representing family, and every family represents at most one element. So $P\varepsilon_A$ takes a collection of families to the set of elements (if any) that they represent.

Now on to understanding $PP^{\mathrm{op}}\pi_1$. Say a *picture* is an element of $P^{\mathrm{op}}(PA \times PB)$, as visualized in Figure 1. Notice that $P^{\mathrm{op}}\pi_1$ is an arrow $P^{\mathrm{op}}PA \to P^{\mathrm{op}}(PA \times PB)$. Remembering again that $P^{(\mathrm{op})}$ on arrows is inverse image, we realize that $P^{\mathrm{op}}\pi_1(F)$ is the picture obtained by including all things which

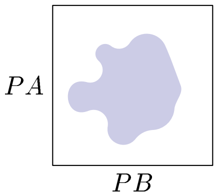

Figure 1: A Picture

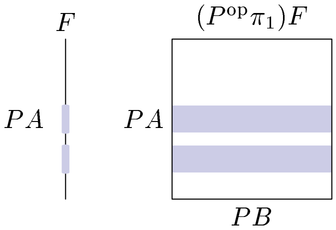

Figure 2: Picture of a Family

project to something in F. See Figure 2. We can apply the same trick as before and say that a picture p represents a family F if $(P^{\mathrm{op}}\pi_1)F = p$, and $PP^{\mathrm{op}}\pi_1$ is then the arrow taking a collection of pictures (call this a *gallery*) to the collection of families represented by any of them.

Now we compose the arrows; Say $h_A := P\epsilon_A PP^{\mathrm{op}}\pi_1$. We see h_A takes a gallery to the collection of elements in A which are represented by a family which is represented by some picture in the gallery. Or we can just say that a picture represents an element if it represents a family representing that element, in which case $P\epsilon_A PP^{\mathrm{op}}\pi_1$ takes a gallery to the collection of elements represented by pictures in it. Then $h := (h_A, h_B)$ is the structure map of the limiting PP^{op}-algebra — it takes a gallery to its *spectrum*, the set of A-elements and B-elements represented by its pictures.

Notice that each element has exactly one picture which represents it. We call these the *canonical pictures*. See Figure 3.

What we've done is found the limit of our diagram hit with K in $\mathbf{Sets}^{PP^{\mathrm{op}}}$. We need to hit things with L again to get back into $\mathbf{Sets}^{\mathrm{op}}$, where it will be the colimit in $\mathbf{Sets}$. Recall that we got L by taking a coequalizer: If we set

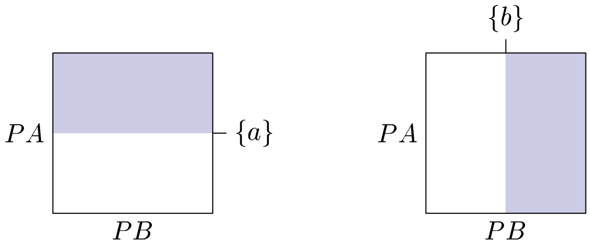

Figure 3: Canonical Pictures

$C := PA \times PB$, it's the coequalizer diagram

$$P^{\mathrm{op}}PP^{\mathrm{op}}C \underset{\varepsilon_{P^{\mathrm{op}}C}}{\overset{P^{\mathrm{op}}h}{\rightrightarrows}} P^{\mathrm{op}}C \overset{e}{\dashrightarrow} L(C,h)$$

in $\mathbf{Sets}^{\mathrm{op}}$. We know we can do this, because it's just an equalizer

$$P^{\mathrm{op}}PP^{\mathrm{op}}C \underset{\varepsilon_{P^{\mathrm{op}}C}}{\overset{P^{\mathrm{op}}h}{\leftleftarrows}} P^{\mathrm{op}}C \overset{e}{\dashleftarrow} L(C,h)$$

in **Sets**—of course **Sets**, being a topos, has finite limits. What's $P^{\mathrm{op}}h$? It takes a subset $S \subseteq PA \times PB$ to the set of galleries whose spectra are in S. But a subset of $PA \times PB$ is a picture; $\varepsilon_{P^{\mathrm{op}}C}$ takes a picture to the collection of galleries that contain it. So the equalizer is going to be a collection of pictures; a picture is in the equalizer if the set of galleries containing it is exactly the set of galleries whose spectra are in it.

Definition A picture p is *good* if, for any gallery G, we have $p \in G \Leftrightarrow h(G) \in p$.

Proposition 3.6 *A picture is good iff it is canonical.*

Proof Suppose p is a canonical picture, say the picture representing $a \in A$. Let a gallery G be given. If $p \in G$, then G has a picture representing a, so at least $a \in \pi_1 h(G)$. But the canonical picture for a includes (α, β) for every $\alpha \ni a$, so we get exactly that $h(G) \in p$. Conversely, if $h(G) \in p$, then G's spectrum is in p; say $h(G) = (\alpha, \beta)$. We infer $a \in \alpha$, so G contains some picture which represents a. But that picture is p, hence $p \in G$.

Conversely, suppose p is good. p cannot be empty; it is contained in the gallery $\{p\}$, so p at least contains the spectrum of $\{p\}$. If p is not canonical, then $h(\{p\}) = \{\emptyset, \emptyset\}$. But this is also the spectrum of the empty gallery, which does not contain p, a contradiction. ∎

3.3 Initial Object

Having thought about binary coproducts so hard, the construction of the initial object is easy. C in this case is 1 instead of $PA \times PB$, (P composed with the unique functor $0 \to \mathbf{Sets}^{\mathrm{op}}$ is the unique functor $0 \to \mathbf{Sets}$, whose limit is the terminal object of $\mathbf{Sets}$) so our "pictures" are now just elements of $P1$, so subsets of 1. The two arrows of the relevant equalizer are

$$P^{\mathrm{op}}PP^{\mathrm{op}}1 \underset{P^{\mathrm{op}}!}{\overset{\varepsilon_{P^{\mathrm{op}}1}}{\leftleftarrows}} P^{\mathrm{op}}1 \dashleftarrow 0$$

Since the codomain of what was h is now 1, our hand is forced to say $h =! : PP^{\mathrm{op}}1 \to 1$. Now we have that any nonempty gallery "represents" (representation is no longer unique) the picture $1 = \{*\}$ and the empty gallery represents the picture $0 = \{\}$. Yet 1 is not contained in the gallery $\{0\}$ which does represent it, and 0 is not contained in the gallery $\{\}$ which represents it. So neither picture is "good", and the equalizer is empty, exactly what we want.