Labelled LF Notes

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1 Language

1.1 Signatures, Contexts

$$\Sigma ::= \cdot \mid \Sigma, c : A \mid \Sigma, a : K$$

$$\Gamma ::= \cdot \mid \Gamma, x : A \mid \Gamma, \alpha : \mathsf{world}$$

1.2 Expressions

$$p, q, r ::= \alpha \mid p \cdot p \mid \epsilon$$

$$K ::= \Pi x{:}A.K \mid \forall \alpha.K \mid \mathsf{type}$$

$$A, B ::= \Pi x{:}A.B \mid a \cdot S \mid \forall \alpha.B \mid {\downarrow}\alpha.B \mid A \mathbin{@} p \mid A \mathbin{\&} B \mid \top$$

$$M, N ::= \lambda x.M \mid R \mid \langle M_1, M_2 \rangle \mid \langle\rangle$$

$$R ::= c \cdot S \mid x \cdot S$$

$$S ::= () \mid (M; S) \mid (\pi_i; S)$$

1.3 Judgments

$$\begin{array}{l}
\Gamma \vdash K : \mathsf{kind} \\
\Gamma \vdash p \Leftarrow \mathsf{world} \\
\Gamma \vdash A : \mathsf{type} \\
\Gamma \vdash S : K > \mathsf{type} \\
\Gamma \vdash M \Leftarrow A[p] \\
\Gamma \vdash R \Rightarrow A[p] \\
\Gamma \vdash S : A[p] > C[r]
\end{array}$$

1.4 Kind Formation

$$\frac{\Gamma \vdash A : \mathsf{type} \qquad \Gamma, x : A \vdash K : \mathsf{kind}}{\Gamma \vdash \Pi x : A.K : \mathsf{kind}} \qquad \frac{\Gamma, \alpha : \mathsf{world} \vdash K : \mathsf{kind}}{\Gamma \vdash \forall \alpha.K : \mathsf{kind}}$$

$$\frac{}{\Gamma \vdash \mathsf{type} : \mathsf{kind}}$$

1.5 World Formation

$$\frac{\alpha : \mathsf{world} \in \Gamma}{\Gamma \vdash \alpha \Leftarrow \mathsf{world}} \qquad \frac{\Gamma \vdash p \Leftarrow \mathsf{world} \qquad \Gamma \vdash q \Leftarrow \mathsf{world}}{\Gamma \vdash p \cdot q \Leftarrow \mathsf{world}} \qquad \frac{}{\Gamma \vdash \epsilon \Leftarrow \mathsf{world}}$$

The relation $\equiv_{ACU}$ holds between two world expressions if they are identical up to associativity and commutativity for $\cdot$, and unit laws for ϵ.

1.6 Type Formation

$$\frac{\Gamma \vdash A : \mathsf{type} \qquad \Gamma, x : A \vdash B : \mathsf{type}}{\Gamma \vdash \Pi x{:}A.B : \mathsf{type}}$$

$$\frac{a : K \in \Sigma \qquad \Gamma \vdash S : K > \mathsf{type}}{\Gamma \vdash a \cdot S : \mathsf{type}}$$

$$\frac{\Gamma, \alpha : \mathsf{world} \vdash B : \mathsf{type}}{\Gamma \vdash \forall \alpha.B : \mathsf{type}}$$

$$\frac{\Gamma, \alpha : \mathsf{world} \vdash B : \mathsf{type}}{\Gamma \vdash \downarrow\alpha.B : \mathsf{type}}$$

$$\frac{\Gamma \vdash A : \mathsf{type} \qquad \Gamma \vdash p \Leftarrow \mathsf{world}}{\Gamma \vdash A \mathbin{@} p : \mathsf{type}}$$

$$\frac{\Gamma \vdash A : \mathsf{type} \qquad \Gamma \vdash B : \mathsf{type}}{\Gamma \vdash A \mathbin{\&} B : \mathsf{type}}$$

$$\frac{}{\Gamma \vdash \top : \mathsf{type}}$$

1.7 Type Checking

$$\frac{\Gamma \vdash R \Rightarrow a \cdot S[p] \qquad S =_\alpha S' \qquad p \equiv_{ACU} q}{\Gamma \vdash R \Leftarrow a \cdot S'[q]}$$

$$\frac{\Gamma, x : A \vdash M \Leftarrow B[p]}{\Gamma \vdash \lambda x.M \Leftarrow \Pi x{:}A.B[p]} \qquad \frac{\Gamma, \alpha : \mathsf{world} \vdash M \Leftarrow B[p]}{\Gamma \vdash M \Leftarrow \forall \alpha.B[p]}$$

$$\frac{\Gamma \vdash M \Leftarrow (\{p/\alpha\}^{\mathsf{world}} B)[p]}{\Gamma \vdash M \Leftarrow \downarrow\alpha.B[p]}$$

$$\frac{\Gamma \vdash M \Leftarrow A[q]}{\Gamma \vdash M \Leftarrow A \mathbin{@} q[p]}$$

$$\frac{\Gamma \vdash M_1 \Leftarrow A_1[p] \qquad \Gamma \vdash M_2 \Leftarrow A_2[p]}{\Gamma \vdash \langle M_1, M_2 \rangle \Leftarrow A_1 \mathbin{\&} A_2[p]}$$

$$\frac{}{\Gamma \vdash \langle\rangle \Leftarrow \top[p]}$$

1.8 Type Synthesis

$$\frac{c : A \in \Sigma \qquad \Gamma \vdash S : A[\epsilon] > C[r]}{\Gamma \vdash c \cdot S \Rightarrow C[r]}$$

$$\frac{x : A \in \Gamma \qquad \Gamma \vdash S : A[\epsilon] > C[r]}{\Gamma \vdash x \cdot S \Rightarrow C[r]}$$

1.9 Type Spine Kinding

$$\frac{}{\Gamma \vdash () : \mathsf{type} > \mathsf{type}}$$

$$\frac{\Gamma \vdash M \Leftarrow A[\epsilon] \qquad \Gamma \vdash S : \{M/x\}^{A} K > \mathsf{type}}{\Gamma \vdash (M; S) : \Pi x{:}A.K > \mathsf{type}}$$

$$\frac{\Gamma \vdash p \Leftarrow \mathsf{world} \qquad \Gamma \vdash S : \{p/\alpha\}^{\mathsf{world}} K > \mathsf{type}}{\Gamma \vdash S : \forall \alpha.K > \mathsf{type}}$$

1.10 Term Spine Typing

$$\frac{}{\Gamma \vdash () : a \cdot S[p] > a \cdot S[p]}$$

$$\frac{\Gamma \vdash M \Leftarrow A[\epsilon] \qquad \Gamma \vdash S : \{M/x\}^{A} B[p] > C[r]}{\Gamma \vdash (M; S) : \Pi x{:}A.B[p] > C[r]}$$

$$\frac{\Gamma \vdash q \Leftarrow \mathsf{world} \qquad \Gamma \vdash S : (\{q/\alpha\}^{\mathsf{world}} B)[p] > C[r]}{\Gamma \vdash S : (\forall \alpha.B)[p] > C[r]}$$

$$\frac{\Gamma \vdash S : \{p/\alpha\}^{\mathsf{world}} B[p] > C[r]}{\Gamma \vdash S : {\downarrow}\alpha.B[p] > C[r]}$$

$$\frac{\Gamma \vdash S : A[q] > C[r]}{\Gamma \vdash S : A \mathbin{@} q[p] > C[r]}$$

$$\frac{\Gamma \vdash S : A_i[p] > C[r]}{\Gamma \vdash (\pi_i; S) : A_1 \mathbin{\&} A_2[p] > C[r]}$$

1.11 Substitution

For uniformity of syntax:

$$\mathsf{x} ::= x \mid \alpha$$
$$\mathsf{X} ::= M \mid p$$
$$\mathsf{U} ::= A \mid \mathsf{world}$$

The following are partial functions on terms yielding terms:

$$\{\mathsf{X}/\mathsf{x}\}^{\mathsf{U}} \qquad [N \mid S]^{B}$$

The first is also overloaded to take kinds to kinds and types to types. Let σ abbreviate $\{\mathsf{X}/\mathsf{x}\}^{\mathsf{U}}$.

1.11.1 Substitution on Kinds

$$\sigma(\Pi x{:}A.K) = \Pi x{:}(\sigma A).(\sigma K)$$
$$\sigma(\forall\alpha.K) = \forall\alpha.(\sigma K)$$
$$\sigma\,\mathsf{type} = \mathsf{type}$$

1.11.2 Substitution on Types

$$\sigma(\Pi x{:}A.B) = \Pi x{:}(\sigma A).(\sigma B)$$
$$\sigma(a \cdot S) = a \cdot (\sigma S)$$
$$\sigma(\forall\alpha.B) = \forall\alpha.(\sigma B)$$
$$\sigma(\downarrow\alpha.B) = \downarrow\alpha.(\sigma B)$$
$$\sigma(A \mathbin{@} p) = (\sigma A) \mathbin{@} (\sigma p)$$
$$\sigma(A \mathbin{\&} B) = \sigma A \mathbin{\&} \sigma B$$
$$\sigma\top = \top$$

1.11.3 Substitution on Terms

$$\sigma(\lambda x.M) = \lambda x.(\sigma M)$$
$$\sigma\langle M_1, M_2\rangle = \langle \sigma M_1, \sigma M_2\rangle$$
$$\sigma\langle\rangle = \langle\rangle$$
$$\sigma\epsilon = \epsilon$$
$$\sigma(p \cdot q) = \sigma p \cdot \sigma q$$
$$\sigma(c \cdot S) = c \cdot (\sigma S)$$
$$\{N/x\}^{C}(x \cdot S) = [N \mid \{N/x\}^{C} S]^{C}$$
$$\sigma(x \cdot S) = x \cdot (\sigma S) \qquad (\text{if } \sigma \text{ not a subst. for } x)$$
$$\{p/\alpha\}^{\mathsf{world}}\alpha = p$$

$$\sigma\alpha = \alpha \qquad \text{(if } \sigma \text{ not a subst. for } \alpha\text{)}$$
$$\sigma() = ()$$
$$\sigma(M; S) = (\sigma M; \sigma S)$$
$$\sigma(\pi_i; S) = (\pi_i; \sigma S)$$

1.12 Reduction

$$[\lambda x.M \mid (N; S)]^{\Pi x:A.B} = [\{N/x\}^A M \mid S]^{\{M/x\}^A B}$$
$$[R \mid ()]^{a \cdot S} = R$$
$$[M \mid S]^{\forall \alpha.A} = [M \mid S]^A$$
$$[M \mid S]^{\downarrow \alpha.A} = [M \mid S]^A$$
$$[M \mid S]^{A @ p} = [M \mid S]^A$$
$$[\langle M_1, M_2\rangle \mid (\pi_i; S)]^{A_1 \& A_2} = [M_i \mid S]^{A_i}$$

Let $\Gamma \vdash M \Leftarrow A$ abbreviate $\Gamma \vdash M \Leftarrow A[\epsilon]$. Let J stand for any judgment.

Lemma 1.1 (Weakening) *If* $\Gamma, \Gamma' \vdash J$, *then* $\Gamma, x : A, \Gamma' \vdash J$.

Say $\mathsf{X}\!\downarrow$ for "X is defined," and $\mathsf{X} = \mathsf{X}'\!\downarrow$ stands for $\mathsf{X}\!\downarrow, \mathsf{X}'\!\downarrow$, and $\mathsf{X} = \mathsf{X}'$.

Lemma 1.2 *Make the abbreviation* $\sigma_i = \{\mathsf{X}_i/\mathsf{x}_i\}^{A_i}$.

1. *Suppose* $\sigma_1\mathsf{X}\!\downarrow$, $\sigma_2\mathsf{X}\!\downarrow$, *and* $\sigma_2\mathsf{X}_1\!\downarrow$ *Suppose* x_1 *does not occur free in* X_2. *Then*
$$\sigma_2\sigma_1\mathsf{X} = \{\sigma_2\mathsf{X}_1/\mathsf{x}_1\}^{A_1}\sigma_2\mathsf{X}\!\downarrow$$

2. *Suppose* $[M|S]^{C^-}\!\downarrow$, $\sigma_1 M\!\downarrow$, *and* $\sigma_1 S\!\downarrow$. *Then*
$$\sigma_1[M|S]^C = [\sigma_1 M|\sigma_1 S]^C\!\downarrow$$

Theorem 1.3 (Substitution) *Assume* Γ, J, A, p *are well-formed.*

- *If* $\Gamma \vdash X \Leftarrow U$ *and* $\Gamma, x : U, \Gamma' \vdash J$ *then* $\Gamma, \sigma\Gamma' \vdash \sigma J$.
- *If* $\Gamma \vdash M \Leftarrow A[p]$ *and* $\Gamma \vdash S : A[p] > C[q]$, *then* $\Gamma \vdash [M \mid S]^A : C[q]$.

Proof By induction. ■

2 Embedding LLF

We claim that a variant of LLF embeds faithfully in the present system with the definition

$$A \multimap B \equiv \forall \alpha . {\downarrow} \beta . (A \mathbin{@} \alpha) \to (B \mathbin{@} (\beta \cdot \alpha))$$

By various inversion properties, one can see that this is admissibly equivalent to adding the two rules

$$\frac{\Gamma, \alpha : \mathsf{world}, x : A \mathbin{@} \alpha \vdash M \Leftarrow B[p \cdot \alpha]}{\Gamma \vdash \lambda x.M \Leftarrow A \multimap B[p]}$$

$$\frac{\Gamma \vdash q \Leftarrow \mathsf{world} \qquad \Gamma \vdash M \Leftarrow A[q] \qquad \Gamma \vdash S : B[p \cdot q] > C[r]}{\Gamma \vdash (M; S) : A \multimap B[p] > C[r]}$$

The variant of LLF we have in mind is one given in canonical forms style, with linear and unrestricted lambda conflated. The syntax of its types is

$$A, B ::= a \cdot S \mid A \multimap B \mid \Pi x{:}A.B \mid A \mathbin{\&} B \mid \top$$

and its term language is identical to HLF's.

Its central typing judgments are:

$$\begin{array}{l} \Gamma; \Delta \vdash_{LLF} M \Leftarrow A \\ \Gamma; \Delta \vdash_{LLF} R \Rightarrow A \\ \Gamma; \Delta \vdash_{LLF} S : A > C \end{array}$$

where Δ is a context of *linear hypotheses* $x \hat{:} A$, subject to exchange but not contraction or weakening.

Its typing rules are

$$\frac{\Gamma; \Delta \vdash_{LLF} R \Rightarrow a \cdot S \qquad S =_\alpha S'}{\Gamma; \Delta \vdash_{LLF} R \Leftarrow a \cdot S'}$$

$$\frac{\Gamma, x : A; \Delta \vdash_{LLF} M \Leftarrow B}{\Gamma; \Delta \vdash_{LLF} \lambda x.M \Leftarrow \Pi x{:}A.B}$$

$$\frac{\Gamma; \Delta, x \hat{:} A \vdash_{LLF} M \Leftarrow B}{\Gamma; \Delta \vdash_{LLF} \lambda x.M \Leftarrow A \multimap B}$$

$$\frac{\Gamma; \Delta \vdash_{LLF} M_1 \Leftarrow A_1 \qquad \Gamma; \Delta \vdash_{LLF} M_2 \Leftarrow A_2}{\Gamma; \Delta \vdash_{LLF} \langle M_1, M_2 \rangle \Leftarrow A_1 \mathbin{\&} A_2}$$

$$\frac{}{\Gamma; \Delta \vdash_{LLF} \langle\rangle \Leftarrow \top}$$

$$\frac{c : A \in \Sigma \qquad \Gamma; \Delta \vdash_{LLF} S : A > C}{\Gamma; \Delta \vdash_{LLF} c \cdot S \Rightarrow C}$$

$$\frac{x:A\in\Gamma \qquad \Gamma;\Delta\vdash_{LLF} S:A>C}{\Gamma;\Delta\vdash_{LLF} x\cdot S\Rightarrow C} \qquad \frac{\Gamma;\Delta\vdash_{LLF} S:A>C}{\Gamma;\Delta,x\hat{:}A\vdash_{LLF} x\cdot S\Rightarrow C}$$

$$\frac{}{\Gamma;\cdot\vdash_{LLF} ():a\cdot S>a\cdot S}$$

$$\frac{\Gamma;\cdot\vdash_{LLF} M\Leftarrow A \qquad \Gamma;\Delta\vdash_{LLF} S:\{M/x\}^A B>C}{\Gamma;\Delta\vdash_{LLF} (M;S):\Pi x{:}A.B>C}$$

$$\frac{\Gamma;\Delta_1\vdash_{LLF} M\Leftarrow A \qquad \Gamma;\Delta_2\vdash_{LLF} S:B>C}{\Gamma;\Delta_1,\Delta_2\vdash_{LLF} (M;S):A\multimap B>C}$$

$$\frac{\Gamma;\Delta\vdash_{LLF} S:A_i>C}{\Gamma;\Delta\vdash_{LLF} (\pi_i;S):A_1\ \&\ A_2>C}$$

Definition Given an LLF context Δ, we define a context $\Delta^@$ as follows:

$$(x_1:A_1,\ldots,x_n:A_n)^@ =$$

$$(\alpha_{x_1}:\mathsf{world},x_1:(A_1\ @\ \alpha_{x_1}),\ldots,\alpha_{x_n}:\mathsf{world},x_n:(A_n\ @\ \alpha_{x_n}))$$

We also define the LLF context

$$\Delta\rfloor_p = (x_{i_1}:A_{i_1},\ldots,x_{i_m}:A_{i_m})$$

whenever $p\equiv_{ACU}\alpha_{x_{i_1}}\cdots\alpha_{x_{i_m}}$ for distinct $i_1,\ldots,i_m$, such that $x_{i_k}:A_{i_k}\in\Delta$ for every $k\in 1\ldots m$.

Remark The definition of $\Delta\rfloor_p$ is well-defined up to commutativity and associativity for world concatenation, because we have exchange and associativity on contexts.

Lemma 2.1 (Factorization) *Suppose A is an LLF type. If $\Gamma,\Delta^@\vdash S : A[p]>C[r]$, then there exists world s such that $r\equiv_{ACU} p\cdot s$.*

Proof By induction on the typing derivation.

Case: Linear spine cons:

$$\frac{\Gamma,\Delta^@\vdash q:\mathsf{world} \qquad \Gamma,\Delta^@\vdash M:A[q] \qquad \Gamma,\Delta^@\vdash S:B[p\cdot q]>C[r]}{\Gamma,\Delta^@\vdash (M;S):A\multimap B[p]>C[r]}$$

By induction hypothesis, r factors as $(p\cdot q)\cdot s'$ for some s'. Therefore by associativity it also factors as $p\cdot(q\cdot s')$. Set $s=q\cdot s'$.

Case: Ordinary spine cons:

$$\frac{\Gamma, \Delta^{@} \vdash M : A[\epsilon] \qquad \Gamma, \Delta^{@} \vdash S : \{M/x\}^A B[p] > C[r]}{\Gamma, \Delta^{@} \vdash (M; S) : \Pi x{:}A.B[p] > C[r]}$$

By induction hypothesis, r factors as $p \cdot s$ for some s, and we done.

Case: Nil:

$$\frac{}{\Gamma, \Delta^{@} \vdash () \vdash a \cdot S[p] > a \cdot S[p]}$$

Set $s = \epsilon$.

Case: Projection:

$$\frac{\Gamma, \Delta^{@} \vdash S : A_i[p] > C[r]}{\Gamma, \Delta^{@} \vdash (\pi_i; S) : A_1 \ \& \ A_2[p] > C[r]}$$

By induction hypothesis, r factors as $p \cdot s$ for some s, and we done.

■

Lemma 2.2 (Soundness) *Suppose Δ is a valid LLF context, and A is a valid LLF type. If $\Gamma, \Delta^{@} \vdash M : A[p]$ (up to permutation of $\Gamma, \Delta^{@}$) then $\Gamma; \Delta\lfloor_p \vdash_{LLF} M : A$.*

Similarly if $\Gamma, \Delta^{@} \vdash S : A[p] > C[r]$ (up to permutation of $\Gamma, \Delta^{@}$) and $r =_{ACU} p \cdot q$, then $\Gamma; \Delta\lfloor_q \vdash_{LLF} S : A > C$.

Proof By induction on the typing derivation.

Case: Linear lambda:

$$\frac{\Gamma, \Delta^{@}, \alpha_x : \mathsf{world}, x : A \ @ \ \alpha_x \vdash M : B[p \cdot \alpha_x]}{\Gamma, \Delta^{@} \vdash \lambda x.M : A \multimap B[p]}$$

By the induction hypothesis, we get $\Gamma; \Delta\lfloor_p, x : A \vdash_{LLF} M : B$. By rule, we get $\Gamma; \Delta\lfloor_p \vdash_{LLF} \lambda x.M : A \multimap B$.

Case: Regular lambda:

$$\frac{\Gamma, \Delta^{@}, x : A \vdash M : B[p]}{\Gamma, \Delta^{@} \vdash \lambda x.M : \Pi x{:}A.B[p]}$$

By the induction hypothesis, we get $\Gamma, x : A; \Delta\lfloor_p \vdash_{LLF} M : B$. By rule, we get $\Gamma; \Delta\lfloor_p \vdash_{LLF} \lambda x.M : \Pi x{:}A.B$.

Case: Linear variable:

$$\frac{x : A \ @ \ \alpha_x \in \Delta^{@} \qquad \dfrac{\Gamma, \Delta^{@} \vdash S : A[\alpha_x] > C[r]}{\Gamma, \Delta^{@} \vdash S : A \ @ \ \alpha_x[\epsilon] > C[r]}}{\Gamma, \Delta^{@} \vdash x \cdot S : C[r]}$$

By lemma, r factors as $q \cdot \alpha_x$ for some q, and by induction hypothesis

$$\Gamma; \Delta\lfloor_q \vdash_{LLF} S : A > C$$

By rule, (since $x : A \mathbin{@} \alpha_x \in \Delta^@$ means we must have had $x : A \in \Delta$)

$$\Gamma; \Delta\lfloor_q, x : A \vdash_{LLF} x \cdot S : C$$

as required, because $\Delta\lfloor_r = \Delta\lfloor_{q \cdot \alpha_x} = \Delta\lfloor_q, x : A$.

Case: Ordinary variable:

$$\frac{x : A \in \Gamma \qquad \Gamma, \Delta^@ \vdash S : A[\epsilon] > C[r]}{\Gamma, \Delta^@ \vdash x \cdot S : C[r]}$$

By induction hypothesis

$$\Gamma; \Delta\lfloor_r \vdash_{LLF} S : A > C$$

By rule,

$$\Gamma; \Delta\lfloor_r \vdash_{LLF} x \cdot S : A > C$$

as required.

Case: Linear spine cons:

$$\frac{\Gamma, \Delta^@ \vdash q : \mathsf{world} \qquad \Gamma, \Delta^@ \vdash M : A[q] \qquad \Gamma, \Delta^@ \vdash S : B[p \cdot q] > C[r]}{\Gamma, \Delta^@ \vdash (M; S) : A \multimap B[p] > C[r]}$$

By lemma, r factors as $p \cdot q \cdot s$ for some s. By induction hypothesis

$$\Gamma; \Delta\lfloor_q \vdash_{LLF} M : A$$

$$\Gamma; \Delta\lfloor_s \vdash_{LLF} S : A > C$$

By assumption $r = p \cdot q'$ for some q'. But we know then that $p \cdot q \cdot s = p \cdot q'$, so by cancellativity of $\cdot$ we infer $q' = q \cdot s$. By rule,

$$\Gamma; \Delta\lfloor_{q \cdot s} \vdash_{LLF} (M; S) : A \multimap B > C$$

as required.

Case: Ordinary spine cons:

$$\frac{\Gamma, \Delta^@ \vdash M : A[\epsilon] \qquad \Gamma, \Delta^@ \vdash S : [M/x]^A B[p] > C[r]}{\Gamma, \Delta^@ \vdash (M; S) : \Pi x{:}A.B[p] > C[r]}$$

Let q be such that $p \cdot q \equiv_{ACU} r$. By induction hypothesis

$$\Gamma; \Delta\lfloor_q \vdash_{LLF} S : [M/x]^A B > C$$

$$\Gamma; \cdot \vdash_{LLF} M : A$$

by rule,

$$\Gamma; \Delta\lfloor_q \vdash_{LLF} (M; S) : \Pi x{:}A.B > C$$

as required.

Case: Nil:

$$\frac{}{\Gamma, \Delta^@ \vdash () : a \cdot S[p] > a \cdot S[p]}$$

The only q such that $p \cdot q = p$ is ϵ. By rule,

$$\Gamma; \cdot \vdash_{LLF} () : a \cdot S > a \cdot S$$

as required.

■

Definition Let α_Δ be the concatenation of all worlds in $\Delta^@$,

XXX should probably state weakening, exchange explicitly

Lemma 2.3 (Completeness) *Suppose all relevant things are valid. (e.g. suppose Γ well-formed, suppose A a valid type in Γ)*

- *If* $\Gamma; \Delta \vdash_{LLF} M \Leftarrow A$, *then* $\Gamma, \Delta^@ \vdash M \Leftarrow A[\alpha_\Delta]$
- *If* $\Gamma; \Delta \vdash_{LLF} R \Rightarrow A$, *then* $\Gamma, \Delta^@ \vdash R \Rightarrow A[\alpha_\Delta]$
- *If* $\Gamma; \Delta \vdash_{LLF} S : A > C$, *then* $\Gamma, \Delta' \vdash S : A[p] > C[q]$, *for any* Δ' *that extends* $\Delta^@$ *and any* p *such that* $\Delta' \vdash p : \mathsf{world}$, *for some* $q \equiv_{ACU} \alpha_\Delta \cdot p$.

Proof By induction on the derivation.

Case:

$$\frac{\Gamma; \Delta \vdash_{LLF} R \Rightarrow a \cdot S \qquad S =_\alpha S'}{\Gamma; \Delta \vdash_{LLF} R \Leftarrow a \cdot S'}$$

Immediate by applying rule to induction hypothesis, since $\alpha_\Delta \equiv_{ACU} \alpha_\Delta$.

Case:

$$\frac{\Gamma, x : A; \Delta \vdash_{LLF} M \Leftarrow B}{\Gamma; \Delta \vdash_{LLF} \lambda x.M \Leftarrow \Pi x{:}A.B}$$

By i.h., $\Gamma, x : A, \Delta^@ \vdash M \Leftarrow B[\alpha_\Delta]$. By exchange and rule application, $\Gamma, \Delta^@ \vdash \lambda x.M \Leftarrow \Pi x{:}A.B[\alpha_\Delta]$.

Case:

$$\frac{\Gamma; \Delta, x \hat{:} A \vdash_{LLF} M \Leftarrow B}{\Gamma; \Delta \vdash_{LLF} \lambda x.M \Leftarrow A \multimap B}$$

By i.h., $\Gamma, \Delta^@, \alpha_x : \mathsf{world}, x : A \mathbin{@} \alpha_x \vdash M \Leftarrow B[\alpha_\Delta \cdot \alpha_x]$. By rule application, $\Gamma, \Delta^@ \vdash \lambda x.M \Leftarrow A \multimap B[\alpha_\Delta]$.

Case:

$$\frac{x : A \in \Gamma \qquad \Gamma; \Delta \vdash_{LLF} S : A > C}{\Gamma; \Delta \vdash_{LLF} x \cdot S \Rightarrow C}$$

By i.h., (choosing $p = \epsilon$) we have $\Gamma, \Delta^@ \vdash S : A[\epsilon] > C[\alpha_\Delta]$. By rule application (since $x : A \in \Gamma$) we get $\Gamma, \Delta^@ \vdash x \cdot S \Rightarrow C[\alpha_\Delta]$.

Case:

$$\frac{\Gamma; \Delta \vdash_{LLF} S : A > C}{\Gamma; \Delta, x \hat{:} A \vdash_{LLF} x \cdot S \Rightarrow C}$$

$\Gamma, \Delta^{@}, \alpha_x : \mathsf{world}, x : A \mathbin{@} \alpha_x \vdash S : A[\alpha_x] > C[\alpha_\Delta \cdot \alpha_x]$ i.h.
$\Gamma, \Delta^{@}, \alpha_x : \mathsf{world}, x : A \mathbin{@} \alpha_x \vdash S : A @ \alpha_x[\epsilon] > C[\alpha_\Delta \cdot \alpha_x]$ by rule
$\Gamma, \Delta^{@}, \alpha_x : \mathsf{world}, x : A \mathbin{@} \alpha_x \vdash x \cdot S \Rightarrow C[\alpha_\Delta \cdot \alpha_x]$ by rule.

Case:

$$\frac{}{\Gamma; \cdot \vdash_{LLF} () : a \cdot S > a \cdot S}$$

Immediate. Here $\alpha_\Delta = \epsilon$, and $\Gamma \vdash () : a \cdot S[p] > a \cdot S[p]$ and $p \cdot \epsilon \equiv_{ACU} p$.

Case:

$$\frac{\Gamma; \cdot \vdash_{LLF} M \Leftarrow A \qquad \Gamma; \Delta \vdash_{LLF} S : \{M/x\}^A B > C}{\Gamma; \Delta \vdash_{LLF} (M; S) : \Pi x{:}A.B > C}$$

Let Δ' extending $\Delta^{@}$ and p such that $\Delta' \vdash p \Leftarrow \mathsf{world}$ be given.

$\Gamma, \Delta' \vdash S : \{M/x\}^A B[p] > C[q]$ by i.h.
$(q \equiv_{ACU} \alpha_\Delta \cdot p)$
$\Gamma \vdash M \Leftarrow A[\epsilon]$ i.h.
$\Gamma, \Delta' \vdash M \Leftarrow A[\epsilon]$ weakening.
$\Gamma, \Delta' \vdash (M; S) : \Pi x{:}A.B[p] > C[q]$ by rule.

Case:

$$\frac{\Gamma; \Delta_1 \vdash_{LLF} M \Leftarrow A \qquad \Gamma; \Delta_2 \vdash_{LLF} S : B > C}{\Gamma; \Delta_1, \Delta_2 \vdash_{LLF} (M; S) : A \multimap B > C}$$

Let Δ' extending $\Delta_1^{@}, \Delta_2^{@}$ and p such that $\Delta' \vdash p \Leftarrow \mathsf{world}$ be given.

$\Gamma, \Delta' \vdash S : \{M/x\}^A B[p \cdot \alpha_{\Delta_1}] > C[q]$ by i.h.
$(q \equiv_{ACU} \alpha_{\Delta_2} \cdot (p \cdot \alpha_{\Delta_1}))$
$\Gamma, \Delta_1^{@} \vdash M \Leftarrow A[\alpha_{\Delta_1}]$ i.h.
$\Gamma, \Delta' \vdash M \Leftarrow A[\alpha_{\Delta_1}]$ weakening.
$\Gamma, \Delta' \vdash (M; S) : A \multimap B[p] > C[q]$ by rule.

■

3 Embedding BI

3.1 BI

Here is the bunched sequent calculus: (XXX need to show that this works vis-a-vis the standard presentation, by eliminating the need for explicit weakening rule!)

$$\frac{}{\Gamma; A \vdash_{BI} A} \qquad \frac{\Gamma(\Delta; \Delta) \vdash_{BI} C}{\Gamma(\Delta) \vdash_{BI} C}$$

$$\frac{\Gamma, A \vdash_{BI} B}{\Gamma \vdash_{BI} A \multimap B} \qquad \frac{\Delta \vdash_{BI} A \qquad \Gamma(B) \vdash_{BI} C}{\Gamma(\Delta, A \multimap B) \vdash_{BI} C}$$

$$\frac{\Gamma; A \vdash_{BI} B}{\Gamma \vdash_{BI} A \rightharpoonup B} \qquad \frac{\Delta \vdash_{BI} A \qquad \Gamma(B) \vdash_{BI} C}{\Gamma(\Delta; A \rightharpoonup B) \vdash_{BI} C}$$

$$\frac{\Gamma \vdash_{BI} A_1 \qquad \Gamma \vdash_{BI} A_2}{\Gamma \vdash_{BI} A_1 \mathbin{\&} A_2} \qquad \frac{\Gamma(A_i) \vdash_{BI} C}{\Gamma(A_1 \mathbin{\&} A_2) \vdash_{BI} C}$$

$$\frac{}{\Gamma \vdash_{BI} \top}$$

3.2 Labellings

Here is an auxiliary data structure that contains both the structural information of a bunched context, and the label information of an HLF context:

$$\text{Labellings} \quad L ::= \cdot \mid (L, L) \mid (L; L) \mid x : A[p]$$

Labellings are identified up to the same sort of associative and commutative laws that bunched contexts respect. Variable names may be shared across additive context joins.

We define below some typing rules that establish whent these two threads of information are compatible.

Here are the two functions $\mathbf{h}, \mathbf{b}$ that project out the HLF and BI information.

$$\begin{aligned} \mathbf{h}(\cdot) &:= \cdot & \mathbf{b}(\cdot) &:= 1_m \\ \mathbf{h}(L_1, L_2) &:= \mathbf{h}(L_1), \mathbf{h}(L_2) & \mathbf{b}(L_1, L_2) &:= \mathbf{b}(L_1), \mathbf{b}(L_2) \\ \mathbf{h}(L_1; L_2) &:= \mathbf{h}(L_1) \cup \mathbf{h}(L_2) & \mathbf{b}(L_1; L_2) &:= \mathbf{b}(L_1); \mathbf{b}(L_2) \\ \mathbf{h}(x : A[p]) &:= x : A \mathbin{@} p & \mathbf{b}(x : A[p]) &:= A \end{aligned}$$

The relation $\mapsto$ takes in a labelling and emits a world that says what resources are required to use the whole context.

$$\frac{}{\cdot \mapsto \epsilon} \qquad \frac{L_1 \mapsto p_1 \qquad L_2 \mapsto p_2}{(L_1, L_2) \mapsto p_1 \cdot p_2} \qquad \frac{L_1 \mapsto p \qquad L_2 \mapsto p}{(L_1; L_2) \mapsto p}$$

Note that already we can see that a labelling that has two ;-subtrees with different resources are somehow 'ill-typed'. Here are the full typing rules, using the notation $L_1(L_2)$ as in BI for a labelling-with-hole L_1 that has L_2 as a subtree occupying that hole:

$$\frac{}{\vec{\alpha} \vdash \cdot : \mathrm{lab}}\,\textsf{unit} \qquad \frac{\vec{\alpha} \vdash L_1(L_2) : \mathrm{lab}}{\vec{\alpha} \vdash L_1(L_2; L_2) : \mathrm{lab}}\,\textsf{contract}$$

$$\frac{\vec{\alpha} \vdash L : \mathrm{lab}}{\vec{\alpha}, \beta \vdash (L, x : A[\beta]) : \mathrm{lab}}\,\textsf{me} \qquad \frac{\vec{\alpha} \vdash L_1(L_2, x : A[p]) : \mathrm{lab} \qquad L_2 \mapsto q}{\vec{\alpha} \vdash L_1(y : B[p \cdot q]) : \mathrm{lab}}\,\textsf{mm}$$

$$\frac{\vec{\alpha} \vdash L : \mathrm{lab} \qquad L \mapsto q}{\vec{\alpha} \vdash (L; x : A[q]) : \mathrm{lab}}\,\mathsf{ae} \qquad \frac{\vec{\alpha} \vdash L_1(L_2; x : A[p]) : \mathrm{lab} \qquad L_2 \mapsto p}{\vec{\alpha} \vdash L_1(y : B[p]) : \mathrm{lab}}\,\mathsf{am}$$

The first character of each two-letter rule means 'additive' or 'multiplicative', and the second character means 'extend' or 'merge'.

3.3 Completeness

Lemma 3.1 *If $x : A \in \Gamma$, then $\Gamma \vdash \eta^*_A(x) : A[\epsilon]$.*

Lemma 3.2 *If $\vec{\alpha} \vdash L_1(L_2) \mapsto q$, and $L_2 \mapsto p$ and $L'_2 \mapsto p$, then $\vec{\alpha} \vdash L_1(L'_2) \mapsto q$.*

Lemma 3.3 *If $\vec{\alpha} \vdash L_1(L_2) : \mathrm{lab}$, then there exists q such that $L_2 \mapsto q$.*

Lemma 3.4 *If $\vec{\alpha} \vdash L : \mathrm{lab}$ and L' is identical to L except that the types in it differ, then so too $\vec{\alpha} \vdash L' : \mathrm{lab}$. Moreover, if $L \mapsto p$, then also $L' \mapsto p$.*

Proposition 3.5 (Completeness) *If $\vec{\alpha} \vdash L : \mathrm{lab}$ and $L \mapsto p$ and $\mathbf{b}(L) \vdash_{BI} A$, then there exists M such that $\vec{\alpha} : \mathsf{world}, \mathbf{h}(L) \vdash M \Leftarrow A[p]$.*

Proof By induction on the derivation of $\mathbf{b}(L) \vdash_{BI} A$.

Case:

$$\frac{}{\Gamma; A \vdash_{BI} A}$$

Since $\mathbf{b}(L) = \Gamma; A$ and $L \mapsto p$, we know L is of the form $L'; x : A[p]$, and so $\mathbf{h}(L) = \mathbf{h}(L') \cup x : A \ @\ p$. Because $x : A \ @\ p \in \mathbf{h}(L)$, it follows from Lemma 3.1 that there is an M such that $\vec{\alpha} : \mathsf{world}, \mathbf{h}(L) \vdash M \Leftarrow A \ @\ p[\epsilon]$. By inversion also $\vec{\alpha} : \mathsf{world}, \mathbf{h}(L) \vdash M \Leftarrow A[p]$ as required. (XXX why is the context valid?)

Case:

$$\frac{\Gamma(\Delta; \Delta) \vdash_{BI} C}{\Gamma(\Delta) \vdash_{BI} C}$$

We're given a labelling L such that $L \mapsto p$ and $\mathbf{b}(L) = \Gamma(\Delta)$. Hence L is of the form $L_1(L_2)$. By the rule $\mathsf{contract}$, $\vec{\alpha} \vdash L_1(L_2; L_2) : \mathrm{lab}$, and it can be seen that $\mathbf{b}(L_1(L_2; L_2)) = \Gamma(\Delta; \Delta)$, and $L_1(L_2; L_2) \mapsto p$, and also $\mathbf{h}(L_1(L_2; L_2)) = \mathbf{h}(L_1(L_2))$. Therefore the induction hypothesis yields $\vec{\alpha} : \mathsf{world}, \mathbf{h}(L_1(L_2)) \vdash M \Leftarrow A[p]$, as required.

Case:

$$\frac{\Gamma, A \vdash_{BI} B}{\Gamma \vdash_{BI} A \multimap B}$$

We're given a labelling L such that $L \mapsto p$ and $\mathbf{b}(L) = \Gamma$. By rule me, we have $\vec{\alpha}, \beta \vdash (L, x : A[\beta]) : \mathrm{lab}$. Noting that $\mathbf{b}(L, x : A[\beta]) = (\Gamma, A)$ and $(L, x : A[\beta]) \mapsto (p \cdot \beta)$, apply the induction hypothesis to obtain $\vec{\alpha}, \beta, \mathbf{h}(L), x : A \ @\ \beta \vdash M \Leftarrow B[p \cdot \beta]$. By rule, $\vec{\alpha}, \mathbf{h}(L) \vdash \lambda x.M \Leftarrow A \multimap B[p]$, as required.

Case:

$$\frac{\Delta \vdash_{BI} A \qquad \Gamma(B) \vdash_{BI} C}{\Gamma(\Delta, A \multimap B) \vdash_{BI} C}$$

We're given a labelling L such that $L \mapsto p$ and $\mathbf{b}(L) = \Gamma(\Delta, A \multimap B)$. Hence L is of the form $L_1(L_2, x : A \multimap B[r])$. Let q be such that $L_2 \mapsto q$ by Lemma 3.3. By the rule mm, we have $\vec{\alpha} \vdash L_1(y : B[r \cdot q]) : \text{lab}$. By induction hypothesis we get M, N such that

$$\vec{\alpha}, \mathbf{h}(L_2) \vdash M \Leftarrow A[q]$$

$$\vec{\alpha}, \mathbf{h}(L_1), y : B \mathbin{@} (r \cdot q) \vdash N \Leftarrow C[p]$$

which can be weakened to

$$\Gamma \vdash M \Leftarrow A[q] \tag{$*$}$$

$$\Gamma, y : B \mathbin{@} (r \cdot q) \vdash N \Leftarrow C[p] \tag{$**$}$$

for $\Gamma = \vec{\alpha}, \mathbf{h}(L_1(L_2, x : A \multimap B[r]))$, and we need to find N' such that

$$\Gamma \vdash N' \Leftarrow C[p]$$

Consider the η-expansion of x at type $A \multimap B$, and note that it is of the form $\lambda z.M'$. By Lemma 3.1

$$\Gamma \vdash \lambda z.M' : A \multimap B \mathbin{@} r[\epsilon]$$

hence by inversion there is a derivation of $\Gamma, \alpha : \mathsf{world}, z : A \mathbin{@} \alpha \vdash M' : B \mathbin{@} (r \cdot \alpha)[\epsilon]$. By rule application from $(*)$ we can get $\Gamma \vdash M \Leftarrow A \mathbin{@} q[\epsilon]$ so substitution of q for α and M for z yields $\Gamma \vdash \{M/z\}^{A@q} M' : B \mathbin{@} (r \cdot q)[\epsilon]$.

Doing one more substitution, for y in $(**)$, we get

$$\Gamma \vdash \{\{M/z\}^{A@q} M'/y\}^{B@(r \cdot q)} N \Leftarrow C[p] \tag{$**$}$$

So let $N' = \{\{M/z\}^{A@q} M'/y\}^{B@(r \cdot q)}$.

Case:

$$\frac{\Gamma; A \vdash_{BI} B}{\Gamma \vdash_{BI} A \rightharpoonup B}$$

We're given a labelling L such that $L \mapsto p$ and $\mathbf{b}(L) = \Gamma$. By rule ae, we have $\vec{\alpha} \vdash (L; x : A[p]) : \text{lab}$. Noting that $\mathbf{b}(L; x : A[p]) = (\Gamma; A)$ and $(L; x : A[p]) \mapsto p$, apply the induction hypothesis to obtain $\vec{\alpha}, \mathbf{h}(L), x : A \mathbin{@} p \vdash M \Leftarrow B[p]$. By rule, $\vec{\alpha}, \mathbf{h}(L) \vdash \lambda x.M \Leftarrow A \rightharpoonup B[p]$, as required.

Case:

$$\frac{\Delta \vdash_{BI} A \qquad \Gamma(B) \vdash_{BI} C}{\Gamma(\Delta; A \rightharpoonup B) \vdash_{BI} C}$$

We're given a labelling L such that $L \mapsto p$ and $\mathbf{b}(L) = \Gamma(\Delta; A \rightharpoonup B)$. Hence L is of the form $L_1(L_2; x : A \rightharpoonup B[r])$. Note that $L_2 \mapsto r$. *XXX we know this by well-formedness of the labelling, right?* By the rule am, we have $\vec{\alpha} \vdash L_1(y : B[r]) : \text{lab}$. *XXX we are swapping in one r subtree for another; need to show that $\mapsto$ is invariant up to this kind of surgery* By induction hypothesis we get M, N such that

$$\vec{\alpha}, \mathbf{h}(L_2) \vdash M \Leftarrow A[r]$$

$$\vec{\alpha}, \mathbf{h}(L_1), y : B \mathbin{@} r \vdash N \Leftarrow C[p]$$

and we need to find N' such that

$$\vec{\alpha}, \mathbf{h}(L_1), \mathbf{h}(L_2), x : A \rightharpoonup B \mathbin{@} r \vdash N' \Leftarrow C[p]$$

So let $N' = \{xM/y\}N$. *XXX two finicky issues here: η expansion, and the flattening action on contexts of $\mathbf{h}$ — just need to imitate the $\multimap$ case*

Case:

$$\frac{\Gamma \vdash_{BI} A_1 \qquad \Gamma \vdash_{BI} A_2}{\Gamma \vdash_{BI} A_1 \mathbin{\&} A_2}$$

We have a labelling L such that $\Gamma = \mathbf{b}(L)$, $L \mapsto p$, $\vec{\alpha} \vdash L : \text{lab}$. By induction hypothesis, there are M_1, M_2 such that $\vec{\alpha} : \mathsf{world}, \mathbf{h}(L) \vdash M_i \Leftarrow A_i[p]$ for $i \in \{1, 2\}$. By rule, $\vec{\alpha} : \mathsf{world}, \mathbf{h}(L) \vdash \langle M_1, M_2 \rangle \Leftarrow A_1 \mathbin{\&} A_2[p]$.

Case:

$$\frac{\Gamma(A_i) \vdash_{BI} C}{\Gamma(A_1 \mathbin{\&} A_2) \vdash_{BI} C}$$

We have a labelling L such that $\Gamma(A_1 \mathbin{\&} A_2) = \mathbf{b}(L)$, $L \mapsto p$, $\vec{\alpha} \vdash L : \text{lab}$. Hence L is of the form $L_0(x : A_1 \mathbin{\&} A_2[q])$. Consider $L' = L_0(y : A_i[q])$. Evidently $\vec{\alpha} \vdash L' : \text{lab}$ and $L' \mapsto p$. By induction hypothesis, there is M such that such that $\vec{\alpha} : \mathsf{world}, \mathbf{h}(L') \vdash M \Leftarrow C[p]$. This can be weakened to

$$\vec{\alpha} : \mathsf{world}, \mathbf{h}(L), y : A_i \mathbin{@} q \vdash M \Leftarrow C[p] \qquad (*)$$

The η-expansion of x at type $A_1 \mathbin{\&} A_2 \mathbin{@} q$ is some pair $\langle N_1, N_2 \rangle$, and by Lemma 3.1,

$$\vec{\alpha} : \mathsf{world}, \mathbf{h}(L) \vdash \langle N_1, N_2 \rangle \Leftarrow A_1 \mathbin{\&} A_2 \mathbin{@} q[\epsilon]$$

By inversion and rule application,

$$\vec{\alpha} : \mathsf{world}, \mathbf{h}(L) \vdash N_i \Leftarrow A_i \mathbin{@} q[\epsilon]$$

Hence we can substitute into $(*)$ and achieve

$$\vec{\alpha} : \mathsf{world}, \mathbf{h}(L) \vdash \{N_i/y\}^{A_i @ q} M \Leftarrow C[p]$$

Case:

$$\frac{}{\Gamma \vdash_{BI} \top}$$

By rule, $\vec{\alpha}, \mathbf{h}(L) \vdash \langle\rangle : \top[p]$.

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3.4 Soundness

Restriction is a partial operation; if none of the clauses below is satisified it is undefined.

$$\begin{aligned}
\cdot\downharpoonleft_\epsilon &= \cdot \\
(L, x : A[\alpha])\downharpoonleft_{p\cdot\alpha} &= (L\downharpoonleft_p), x : A[\alpha] \\
(L, x : A[\alpha])\downharpoonleft_p &= (L\downharpoonleft_p) \qquad (\alpha \notin p) \\
(L; x : A[p])\downharpoonleft_p &= (L\downharpoonleft_p), x : A[p] \\
(L; x : A[p])\downharpoonleft_q &= (L\downharpoonleft_p), x : A[q] \qquad (p \neq q)
\end{aligned}$$

We say $\vec\alpha \vdash L : \mathrm{lab}^*$ if there is a proof of $\vec\alpha \vdash L : \mathrm{lab}$ using only rules $\mathsf{u}, \mathsf{me}, \mathsf{ae}$. Say $p \leq q$ if there exists r such that $p \cdot r = q$.

Proposition 3.6 (Soundness) *Suppose $\vec\alpha \vdash L : \mathrm{lab}^*$. Suppose $L \mapsto q$.*

- *If $\Gamma \vdash M \Leftarrow A[p]$, and $\Gamma\downharpoonleft_p = \mathbf{h}(L)$, and $p \leq q$, then $\mathbf{b}(L) \vdash_{BI} A$.*
- *If $\vec\alpha : \mathsf{world}, \mathbf{h}(L) \vdash S \Leftarrow A[p] > C[r]$, and $p, r \leq q$, then $\mathbf{b}(L\downharpoonleft_p) \vdash_{BI} A$ implies $\mathbf{b}(L\downharpoonleft_r) \vdash_{BI} C$.*

Proof By induction on the typing derivation. The cases for the additive pair and unit are easy. The remaining cases are introductions and eliminations for $\rightharpoonup, \multimap$, the variable case, and the nil spine case.

Case:

$$\dfrac{x : A \mathbin{@} p \in \mathbf{h}(L) \quad \dfrac{\overset{\mathcal{D}}{\vec\alpha, \mathbf{h}(L) \vdash S : A[p] > C[r]}}{\vec\alpha, \mathbf{h}(L) \vdash S : A \mathbin{@} p[\epsilon] > C[r]}}{\vec\alpha, \mathbf{h}(L) \vdash x \cdot S \Leftarrow C[r]}$$

From $x : A \mathbin{@} p \in \mathbf{h}L$ we know that $x : A[p] \in L$, which by Lemma XXX gives $\mathbf{b}(L\downharpoonleft_p) \vdash A$. By the induction hypothesis on $\mathcal{D}$, we conclude $\mathbf{b}(L\downharpoonleft_r) \vdash C$, as required.

Case:

$$\dfrac{\dfrac{\dfrac{\dfrac{\overset{\mathcal{D}}{\vec\alpha, \beta, \mathbf{h}(L), x : A \mathbin{@} \beta \vdash M \Leftarrow B[p\cdot\beta]}}{\vec\alpha, \beta, \mathbf{h}(L), x : A \mathbin{@} \beta \vdash M \Leftarrow B \mathbin{@} (p\cdot\beta)[p]}}{\vec\alpha, \beta, \mathbf{h}(L) \vdash \lambda x.M \Leftarrow A \mathbin{@} \beta \to B \mathbin{@} (p\cdot\beta)[p]}}{\vec\alpha, \mathbf{h}(L) \vdash \lambda x.M \Leftarrow \forall\beta.A \mathbin{@} \beta \to B \mathbin{@} (p\cdot\beta)[p]}}{\vec\alpha, \mathbf{h}(L) \vdash \lambda x.M \Leftarrow A \multimap B[p]}$$

Because $(\mathbf{h}(L), x : A \mathbin{@} \beta) = \mathbf{h}(L, x : A[\beta])$ and $(L, x : A[\beta])\downharpoonleft_{p\beta} = (L\downharpoonleft_p), x : A[\beta]$, the induction hypothesis can be applied to $\mathcal{D}$, yielding $\mathbf{b}(L\downharpoonleft_p), A \vdash B$. By rule, $\mathbf{b}(L\downharpoonleft_p) \vdash A \multimap B$, as required.

Case:

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}\\ \vec{\alpha}, \mathbf{h}(L), x : A \mathbin{@} p \vdash M \Leftarrow B[p]\end{array}}{\vec{\alpha}, \mathbf{h}(L) \vdash \lambda x.M \Leftarrow A \mathbin{@} p \rightarrow B[p]}}{\vec{\alpha}, \mathbf{h}(L) \vdash \lambda x.M \Leftarrow A \rightharpoonup B[p]}$$

Because $(\mathbf{h}(L), x : A \mathbin{@} \beta) = \mathbf{h}(L, x : A[\beta])$ and $(L, x : A[\beta])\lfloor_{p\beta} = (L\lfloor_p), x : A[\beta]$, the induction hypothesis can be applied to $\mathcal{D}$, yielding $\mathbf{b}(L\lfloor_p), A \vdash B$. By rule, $\mathbf{b}(L\lfloor_p) \vdash A \multimap B$, as required.

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$$\begin{aligned}
\{A/p\}\cdot &= (\cdot; y : A[\epsilon])\\
\{A/p\alpha\}(L, x : B[\alpha]) &= ((\{X_\alpha \multimap A/p\}L), x : B \mathbin{\&} X_\alpha[\alpha])\\
\{A/p\}(L, x : B[\alpha]) &= ((\{A/p\}L), x : B[\alpha]) \qquad (\alpha \notin p)\\
\{A/p\}(L; x : B[q]) &= ((\{A/p\}L); x : B[q])
\end{aligned}$$

$$\dfrac{\Gamma \vdash M : A[p] \qquad \Gamma \vdash S : B[p] > C[r]}{\Gamma \vdash (M; S) : A \rightharpoonup B[p] > C[r]}$$

Here i.h. gives us $\mathbf{b}(L\lfloor_p) \vdash A$, and $\mathbf{b}(L\lfloor_p) \vdash B$ implies $\mathbf{b}(L\lfloor_r) \vdash C$. We assume $\mathbf{b}(L\lfloor_p) \vdash A \rightharpoonup B$ and try to show $\mathbf{b}(L\lfloor_p) \vdash B$. But this is fairly easy!

$$\dfrac{\Gamma \vdash M : A[q] \qquad \Gamma \vdash S : B[pq] > C[r]}{\Gamma \vdash (M; S) : A \multimap B[p] > C[r]}$$

Here i.h. gives us $\mathbf{b}(L\lfloor_q) \vdash A$, and $\mathbf{b}(L\lfloor_{pq}) \vdash B$ implies $\mathbf{b}(L\lfloor_r) \vdash C$. We assume $\mathbf{b}(L\lfloor_p) \vdash A \multimap B$ and try to show $\mathbf{b}(L\lfloor_p q) \vdash B$. We can get that $\mathbf{b}(L\lfloor_p, L\lfloor_q) \vdash A \otimes (A \multimap B)$ and cut to get $\mathbf{b}(L\lfloor_p, L\lfloor_q) \vdash B$. All we need is that $\mathbf{b}(L\lfloor_{pq})$ is a stronger context than $\mathbf{b}(L\lfloor_p, L\lfloor_q)$, but this is true.