Translating Focusing into Ordered Logic

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Make up two atoms $\triangleleft$ and $\triangleright$.

$$A = v^+F^+ \mid v^-F^-$$
$$F^+ = F^+ \otimes F^+ \mid d^+A$$
$$F^- = F^+ \multimap F^- \mid d^-A$$

Define $\overleftarrow{X}$ and $\overleftarrow{\overleftarrow{A}}$ and $\overrightarrow{X}$ and $\overrightarrow{\overrightarrow{A}}$:

$$\overleftarrow{\overleftarrow{A}} = !(\triangleleft\triangleright \twoheadrightarrow \overleftarrow{A})$$
$$\overrightarrow{\overrightarrow{A}} = \triangleleft\triangleright \bullet !\overrightarrow{A}$$

X	$\overleftarrow{X}$	$\overrightarrow{X}$
v^+F^+	$\triangleleft \bullet \overleftarrow{F^+} \bullet \triangleright$	$\overrightarrow{F^+}$
v^-F^-	$\overleftarrow{F^-}$	$\triangleleft \twoheadrightarrow \overrightarrow{F^-}$
d^+A	$\triangleright \twoheadrightarrow (\triangleright \bullet \overleftarrow{\overleftarrow{A}})$	$!(\triangleleft\triangleright \twoheadrightarrow \overrightarrow{\overrightarrow{A}})$
d^-A	$\overleftarrow{\overleftarrow{A}} \bullet \triangleleft\triangleright$	$\triangleright \twoheadrightarrow \overrightarrow{\overrightarrow{A}}$
$F_1^+ \otimes F_2^+$	$\triangleright \twoheadrightarrow (\overleftarrow{F_1^+} \bullet \overleftarrow{F_2^+} \bullet \triangleright)$	$\overrightarrow{F_1^+} \bullet \overrightarrow{F_2^+}$
$F^+ \multimap F^-$	$\overrightarrow{F^+} \twoheadrightarrow \overleftarrow{F^-}$	$\overleftarrow{F^+} \twoheadrightarrow \overrightarrow{F_-}$

Now prove:

Lemma 0.1 (Identity Lemma)

1. $\overleftarrow{F^+} \bullet \triangleright \vdash \triangleright \bullet !\overrightarrow{F^+}$
2. $\triangleright \twoheadrightarrow \overleftarrow{F^-} \vdash \overrightarrow{F^-}$
3. $\triangleleft\triangleright \twoheadrightarrow \overleftarrow{A}; \triangleleft\triangleright \vdash \overrightarrow{\overrightarrow{A}}$

Proof By induction on the proposition.

1. d^+ case:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\triangleright \vdash \triangleright} \quad \dfrac{\dfrac{\dfrac{\text{Part 3}}{\lhd\rhd \twoheadrightarrow \overleftarrow{A}; \lhd\rhd \vdash \overrightarrow{A}}}{\lhd\rhd \twoheadrightarrow \overleftarrow{A}; \cdot \vdash !(\lhd\rhd \twoheadrightarrow \overrightarrow{A})}}{\overleftarrow{A} \vdash !(\lhd\rhd \twoheadrightarrow \overrightarrow{A})}}{\triangleright \bullet \overleftarrow{A} \vdash \triangleright \bullet !(\lhd\rhd \twoheadrightarrow \overrightarrow{A})}}{\triangleright \twoheadrightarrow (\triangleright \bullet \overleftarrow{A}), \triangleright \vdash \triangleright \bullet !(\lhd\rhd \twoheadrightarrow \overrightarrow{A})}$$

$\otimes$ case:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\triangleright, !\overrightarrow{F_1^+}, !\overrightarrow{F_2^+}, \vdash \triangleright \bullet !(\overrightarrow{F_1^+} \bullet \overrightarrow{F_2^+})}}{\overleftarrow{F_1^+}, \triangleright, !\overrightarrow{F_2^+}, \vdash \triangleright \bullet !(\overrightarrow{F_1^+} \bullet \overrightarrow{F_2^+})}\ \text{Part 1, cut}}{\overleftarrow{F_1^+}, \overleftarrow{F_2^+}, \triangleright \vdash \triangleright \bullet !(\overrightarrow{F_1^+} \bullet \overrightarrow{F_2^+})}\ \text{Part 1, cut}}{\triangleright \twoheadrightarrow (\overleftarrow{F_1^+} \bullet \overleftarrow{F_2^+} \bullet \triangleright), \triangleright \vdash \triangleright \bullet !(\overrightarrow{F_1^+} \bullet \overrightarrow{F_2^+})}$$

2. d^- case:

$$\dfrac{\dfrac{\dfrac{\text{Part 3}}{\overleftarrow{A}, \lhd\rhd \vdash \overrightarrow{A}}}{(\overleftarrow{A} \bullet \lhd\rhd) \vdash \overrightarrow{A}}}{\triangleright \twoheadrightarrow (\lhd\rhd \bullet \overleftarrow{A}) \vdash \triangleright \twoheadrightarrow \overrightarrow{A}}$$

$\multimap$ case:

$$\dfrac{\dfrac{\dfrac{\dfrac{\overrightarrow{F^+} \twoheadrightarrow \overleftarrow{F^-}, \overrightarrow{F^+} \vdash \overleftarrow{F_-}}{\triangleright \twoheadrightarrow (\overrightarrow{F^+} \twoheadrightarrow \overleftarrow{F^-}), \triangleright, \overrightarrow{F^+} \vdash \overleftarrow{F_-}}\ \text{Part 1, cut}}{\triangleright \twoheadrightarrow (\overrightarrow{F^+} \twoheadrightarrow \overleftarrow{F^-}), \overleftarrow{F^+}, \triangleright \vdash \overleftarrow{F_-}}\ \text{Part 1, cut}}{\triangleright \twoheadrightarrow (\overrightarrow{F^+} \twoheadrightarrow \overleftarrow{F^-}), \overleftarrow{F^+} \vdash \overrightarrow{F_-}}\ \text{Part 2, cut}}{\triangleright \twoheadrightarrow (\overrightarrow{F^+} \twoheadrightarrow \overleftarrow{F^-}) \vdash \overleftarrow{F^+} \twoheadrightarrow \overrightarrow{F_-}}$$

3. v^+F^+ case:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\text{Part 1}}{\overrightarrow{\overrightarrow{F^+}}, \triangleright \vdash \triangleright \bullet !\overrightarrow{\overrightarrow{F^+}}} \quad \dfrac{\dfrac{}{\triangleleft, \triangleright, !\overrightarrow{\overrightarrow{F^+}} \vdash \triangleleft\triangleright \bullet !\overrightarrow{\overrightarrow{F^+}}}}{\triangleleft, \triangleright, !\overrightarrow{\overrightarrow{F^+}} \vdash \overrightarrow{\overrightarrow{v^+F^+}}}}{\triangleleft, \overleftarrow{\overleftarrow{F^+}}, \triangleright \vdash \overrightarrow{\overrightarrow{v^+F^+}}}\,\text{cut}}{\overleftarrow{\overleftarrow{v^+F^+}} \vdash \overrightarrow{\overrightarrow{v^+F^+}}}}{\triangleleft\triangleright \twoheadrightarrow \overleftarrow{\overleftarrow{v^+F^+}}; \triangleleft\triangleright \vdash \overrightarrow{\overrightarrow{v^+F^+}}}}{\overleftarrow{\overleftarrow{v^+F^+}}, \triangleleft\triangleright \vdash \overrightarrow{\overrightarrow{v^+F^+}}}$$

v^-F^- case:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\overleftarrow{\overleftarrow{F^-}} \vdash \overleftarrow{\overleftarrow{F^-}}}}{\overleftarrow{\overleftarrow{v^-F^-}}, \triangleleft, \triangleright \vdash \overleftarrow{F^-}}}{\overleftarrow{\overleftarrow{v^-F^-}}, \triangleleft \vdash \triangleright \twoheadrightarrow \overleftarrow{F^-}} \quad \dfrac{}{\triangleright \twoheadrightarrow \overleftarrow{\overleftarrow{F^-}} \vdash \overrightarrow{\overrightarrow{F^-}}}\,\text{cut}}{\overleftarrow{\overleftarrow{v^-F^-}}, \triangleleft \vdash \overrightarrow{F^-}}}{\overleftarrow{\overleftarrow{v^-F^-}} \vdash \overrightarrow{v^-F^-}}}{\overleftarrow{\overleftarrow{v^-F^-}}; \cdot \vdash !\overrightarrow{v^-F^-}}}{\overleftarrow{\overleftarrow{v^-F^-}}; \triangleleft\triangleright \vdash \triangleleft\triangleright \bullet !\overrightarrow{v^-F^-}}$$

Lemma 0.2 (Cut Lemma)

1. $\triangleright \bullet \overrightarrow{F^+} \vdash \overleftarrow{F^+} \bullet \triangleright$
2. $\overrightarrow{F^-} \vdash \triangleright \twoheadrightarrow \overleftarrow{F^-}$
3. $\overrightarrow{A}; \triangleleft\triangleright \vdash \overleftarrow{A}$.

Proof By induction on the proposition.

1. d^+ case:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\text{Part 3}}{\overrightarrow{\overrightarrow{A}};\triangleleft\triangleright\vdash\overleftarrow{\overrightarrow{A}}}}{\overrightarrow{\overrightarrow{A}}\vdash\overleftarrow{\overrightarrow{A}}}}{\triangleleft\triangleright\twoheadrightarrow\overrightarrow{\overrightarrow{A}};\triangleleft\triangleright\vdash\overleftarrow{\overrightarrow{A}}}}{\triangleleft\triangleright\twoheadrightarrow\overrightarrow{\overrightarrow{A}};\cdot\vdash\overleftarrow{\overleftarrow{A}}}}{\triangleleft\triangleright\twoheadrightarrow\overrightarrow{\overrightarrow{A}};\cdot\vdash\triangleright\twoheadrightarrow(\triangleright\bullet\overleftarrow{\overleftarrow{A}})}}{\triangleright,!(\triangleleft\triangleright\twoheadrightarrow\overrightarrow{\overrightarrow{A}})\vdash(\triangleright\twoheadrightarrow(\triangleright\bullet\overleftarrow{\overleftarrow{A}}))\bullet\triangleright}}{\triangleright,\overrightarrow{d^+A}\vdash\overleftarrow{d^+A}\bullet\triangleright}$$

$\otimes$ case:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overleftarrow{F_1^+},\overleftarrow{F_2^+},\triangleright\vdash\overleftarrow{F_1^+}\bullet\overleftarrow{F_2^+}\bullet\triangleright}{\overleftarrow{F_1^+},\overleftarrow{F_2^+}\vdash(\triangleright\twoheadrightarrow(\overleftarrow{F_1^+}\bullet\overleftarrow{F_2^+}\bullet\triangleright))}}{\overleftarrow{F_1^+},\overleftarrow{F_2^+},\triangleright\vdash(\triangleright\twoheadrightarrow(\overleftarrow{F_1^+}\bullet\overleftarrow{F_2^+}\bullet\triangleright))\bullet\triangleright}}{\overleftarrow{F_1^+},\triangleright,\overrightarrow{F_2^+}\vdash(\triangleright\twoheadrightarrow(\overleftarrow{F_1^+}\bullet\overleftarrow{F_2^+}\bullet\triangleright))\bullet\triangleright}\ \text{Part 1, cut}}{\triangleright,\overrightarrow{F_1^+},\overrightarrow{F_2^+}\vdash(\triangleright\twoheadrightarrow(\overleftarrow{F_1^+}\bullet\overleftarrow{F_2^+}\bullet\triangleright))\bullet\triangleright}\ \text{Part 1, cut}$$

2. d^- case:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\text{Part 3}}{\overrightarrow{\overrightarrow{A}};\triangleleft\triangleright\vdash\overleftarrow{\overrightarrow{A}}}}{\overrightarrow{\overrightarrow{A}};\cdot\vdash\overleftarrow{\overleftarrow{A}}}}{\overrightarrow{\overrightarrow{A}};\triangleleft\triangleright\vdash\overleftarrow{\overleftarrow{A}}\bullet\triangleleft\triangleright}}{\overrightarrow{\overrightarrow{A}}\vdash\overleftarrow{\overleftarrow{A}}\bullet\triangleleft\triangleright}}{\triangleright\twoheadrightarrow\overrightarrow{\overrightarrow{A}}\vdash\triangleright\twoheadrightarrow(\overleftarrow{\overleftarrow{A}}\bullet\triangleleft\triangleright)}$$

$\multimap$ case:

$$\dfrac{\dfrac{\dfrac{\overleftarrow{F^+}\twoheadrightarrow\overrightarrow{F_-},\overleftarrow{F^+}\vdash\overrightarrow{F^-}}{\overleftarrow{F^+}\twoheadrightarrow\overrightarrow{F_-},\overleftarrow{F^+},\triangleright\vdash\overleftarrow{F^-}}\ \text{Part 2, cut}}{\overleftarrow{F^+}\twoheadrightarrow\overrightarrow{F_-},\triangleright,\overrightarrow{F^+}\vdash\overleftarrow{F^-}}\ \text{Part 1, cut}}{\overleftarrow{F^+}\twoheadrightarrow\overrightarrow{F_-}\vdash\triangleright\twoheadrightarrow(\overrightarrow{F^+}\twoheadrightarrow\overleftarrow{F^-})}$$

3. v^+F^+ case:

$$\dfrac{\dfrac{}{\lhd \vdash \lhd} \quad \dfrac{\text{Part 1}}{\overrightarrow{\rhd, F^+} \vdash \overleftarrow{F^+} \bullet \rhd}}{\overrightarrow{F^+}; \lhd\!\rhd \vdash \lhd \bullet \overleftarrow{F^+} \bullet \rhd}}{\overrightarrow{v^+F^+}; \lhd\!\rhd \vdash \overleftarrow{v^+F^+}}$$

v^-F^- case:

$$\dfrac{\dfrac{\dfrac{\text{Part 2}}{\overrightarrow{F^-}, \rhd \vdash \overleftarrow{F^-}}}{\lhd \twoheadrightarrow \overrightarrow{F^-}; \lhd\!\rhd \vdash \overleftarrow{F^-}}}{\overrightarrow{v^-F^-}; \lhd\!\rhd \vdash \overleftarrow{v^-F^-}}$$

Corollary 0.3 (Identity) $\overleftarrow{A}, \lhd\!\rhd \vdash \overrightarrow{A}$

Corollary 0.4 (Cut) *If* $\overleftarrow{\Gamma}; \lhd\!\rhd \vdash \overrightarrow{A}$ *and* $\overleftarrow{\Gamma}, \overleftarrow{A}; \lhd\!\rhd \vdash C$*, then* $\overleftarrow{\Gamma}; \lhd\!\rhd \vdash C$*.*

Proof

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\text{Part 3}}{\overrightarrow{A}; \lhd\!\rhd \vdash \overleftarrow{A}}}{\overrightarrow{A} \vdash \overleftarrow{A}}}{\lhd\!\rhd \twoheadrightarrow \overrightarrow{A}, \lhd\!\rhd \vdash \overleftarrow{A}}}{(\lhd\!\rhd \twoheadrightarrow \overrightarrow{A}) \vdash \overleftarrow{A}}}{!(\lhd\!\rhd \twoheadrightarrow \overrightarrow{A}), \lhd\!\rhd \vdash \lhd\!\rhd \bullet !\overleftarrow{A}} \quad \dfrac{}{\lhd\!\rhd \vdash \lhd\!\rhd}$$

By a pair of cuts in ordered logic, we obtain the desired result. ■