Tethered Interval Semantics of Ordered Logic

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1 Introduction

I notice there's an interesting translation that explains ordered logic [PP99] as arising from a logic that can talk about linear assumptions that behave like *intervals* on an abstract line. This translation is a 'syntactic semantics' in the sense that it's a translation whose source and destination languages admit a complete focusing proof strategy, and the translation preserves not just provability, but focused proof search behavior.

I certainly would not claim that this semantics is unequivocally *better* than setting up a resource semantics where every ordered assumption is labelled by an element of a (crucially not assumed to be commutative) monoid, but it's interesting that ordered logic can be 'implemented' in more than one way. This explanation, in terms of the behavior of intervals and concatenation, does have the salient advantage, however, that associativity arises somewhat naturally, instead of needing to be imposed. I think it may for similar reasons more easily generalize to hypergraph-like relations on more than 2 endpoints, or higher-dimensional cells.

2 A Sketch of the Semantics

Consider the language of ordered logic:

$$\begin{array}{lcl} \text{Propositions } A & ::= & p \mid A \twoheadrightarrow B \mid A \rightarrowtail B \mid 1 \mid A \bullet B \mid \\ & & 0 \mid A \oplus B \mid \top \mid A \,\&\, B \end{array}$$

We have atomic propositions p and all the usual connectives, multiplicative and additive.

We take a stab at how to interpret this language into constructive first-order logic with equality. We say that every ordered logic proposition is to be interpreted as a binary relation on some Kripke-like syntactic sort of worlds. We require for every atom p a base relation $\eta_p(u, v)$ on worlds be provided, and then we can define A^u_v, a function whose inputs are an ordered language proposition A and two worlds u and v, and which outputs a first-order proposition. It is defined by recursion on A:

$$(A \twoheadrightarrow B)^u_v = \forall x. A^v_x \Rightarrow B^u_x$$

$$(A \rightarrowtail B)^u_v = \forall x. A^x_u \Rightarrow B^x_v$$

$$1^u_v = (u = v) \qquad (A \bullet B)^u_v = \exists x. A^u_x \wedge B^x_v$$

$$0^u_v = \bot \qquad (A \oplus B)^u_v = A^u_v \vee B^u_v$$

$$\top^u_v = \top \qquad (A \,\&\, B)^u_v = A^u_v \wedge B^u_v$$

$$p^u_v = \eta_p(u, v)$$

We can think of A^u_v as meaning 'A is true in the interval from u to v'.

A context $\Omega = A_1, \ldots, A_n$ is interpreted by saying

$$\Omega^{u_0}_{u_n} = (A_1 \bullet \cdots \bullet A_n)^{u_0}_{u_n}$$
$$= \exists u_1 \cdots u_{n-1}. (A_1)^{u_0}_{u_1} \wedge \cdots \wedge (A_n)^{u_{n-1}}_{u_n}$$

and we say the meaning of a sequent $\Omega \vdash A$ is the closed first-order entailment

$$\vdash \forall uv. \Omega^u_v \Rightarrow A^u_v$$

It is straightforward to prove that this interpretation is sound.

Lemma 2.1 *If $\Omega \vdash A$, then $\vdash \forall uv. \Omega^u_v \Rightarrow A^u_v$.*

Proof Asynchronous rules pretty much come for free, since by design we have rigged things so that the top of each asynchronous rule has the same interpretation as the bottom. For example, let's look at $\twoheadrightarrow R$:

$$\frac{\Omega, A \twoheadrightarrow B}{\Omega \vdash A \twoheadrightarrow B}$$

Suppose $\Omega = A_1, \ldots, A_n$. Then the interpretation of the top of the rule is the first-order provability

$$\vdash \forall u_0 v. (\exists u_n. \Omega^{u_0}_{u_n} \wedge (A)^{u_n}_v) \Rightarrow B^{u_0}_v$$

which is the same thing (by right-invertibility of first-order connectives like $\Rightarrow$ and $\forall$ and left-invertibility of $\wedge$ and $\exists$) as requiring

$$(A_1)^{u_0}_{u_1}, \ldots, (A_n)^{u_{n-1}}_{u_n}, (A)^{u_n}_v \vdash B^{u_0}_v \qquad (*)$$

and the bottom of the rule means

$$\vdash \forall u_0 u_n. \Omega^{u_0}_{u_n} \Rightarrow (\forall v. A^{u_n}_v \Rightarrow B^{u_0}_v)$$

also means the same thing as $(*)$.

Let's now turn to the synchronous rules, starting with $\twoheadrightarrow L$.

$$\frac{\Omega_2 \vdash A \qquad \Omega_1, B, \Omega_3 \vdash C}{\Omega_1, A \twoheadrightarrow B, \Omega_2, \Omega_3 \vdash C}$$

By induction hypothesis, we know $(\Omega_2)^w_y \vdash A^w_y$ and $(\Omega_1)^u_v, B^v_y, (\Omega_3)^y_z \vdash C^u_z$. From them we can derive

$$\cfrac{\cfrac{(\Omega_2)^w_y \vdash A^w_y \qquad (\Omega_1)^u_v, B^v_y, (\Omega_3)^y_z \vdash C^u_z}{(\Omega_1)^u_v, A^w_y \Rightarrow B^v_y, (\Omega_2)^w_y, (\Omega_3)^y_z \vdash C^u_z}}{(\Omega_1)^u_v, \forall x. A^w_x \Rightarrow B^v_x, (\Omega_2)^w_y, (\Omega_3)^y_z \vdash C^u_z}$$

And $\rightarrowtail$ and $\bullet$ go pretty similarly. ■

2.1 Counterexample to Completeness

Unfortunately, this semantics is not complete. A shibboleth sequent distinguishing them is:

$$A \rightarrowtail B \twoheadrightarrow 0, B \vdash (A \rightarrowtail 0) \bullet B$$

First let's see why it fails in real ordered logic. We assume focusing to make the proof of underivability easier. Assume A and B are positive atoms. So focus on $A \rightarrowtail B \twoheadrightarrow 0$ fails because we don't have A yet. We must focus on the right, so we have to consume the B. Then we have the goal $A, A \rightarrowtail B \twoheadrightarrow 0 \vdash 0$, and although we can capture the A, we have no B.

On the other hand, we can prove

$$\frac{\dfrac{}{B^v_w \vdash B^v_w} \qquad \dfrac{}{0^t_w \vdash 0^t_v}\ {\color{blue}0L}}{B^v_w \Rightarrow 0^t_w, B^v_w \vdash 0^t_v}$$

$$\frac{B^v_w \Rightarrow 0^t_w, B^v_w \vdash 0^t_v}{\forall x. B^v_x \Rightarrow 0^t_x, B^v_w \vdash 0^t_v}\ {\color{blue}\forall L}$$

$$\frac{\forall x. B^v_x \Rightarrow 0^t_x, B^v_w \vdash 0^t_v}{(B \twoheadrightarrow 0)^t_v, B^v_w \vdash 0^t_v}$$

$$\frac{(B \twoheadrightarrow 0)^t_v, B^v_w \vdash 0^t_v \qquad \dfrac{}{A^t_u \vdash A^t_u}}{A^t_u, (A \rightarrowtail B \twoheadrightarrow 0)^u_v, B^v_w \vdash 0^t_v}$$

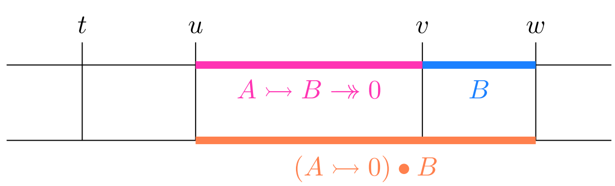
$$\frac{A^t_u, (A \rightarrowtail B \twoheadrightarrow 0)^u_v, B^v_w \vdash 0^t_v}{(A \rightarrowtail B \twoheadrightarrow 0)^u_v, B^v_w \vdash (A \rightarrowtail 0)^u_v}$$

$$\frac{(A \rightarrowtail B \twoheadrightarrow 0)^u_v, B^v_w \vdash (A \rightarrowtail 0)^u_v \qquad \dfrac{}{B^v_w \vdash B^v_w}}{(A \rightarrowtail B \twoheadrightarrow 0)^u_v, B^v_w \vdash (A \rightarrowtail 0)^u_v \wedge B^v_w}$$

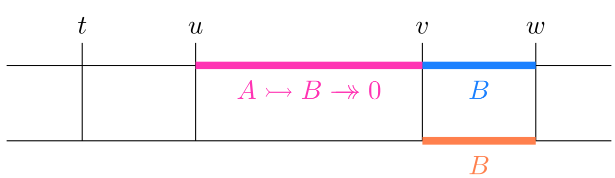
$$\frac{(A \rightarrowtail B \twoheadrightarrow 0)^u_v, B^v_w \vdash (A \rightarrowtail 0)^u_v \wedge B^v_w}{(A \rightarrowtail B \twoheadrightarrow 0)^u_v, B^v_w \vdash ((A \rightarrowtail 0) \bullet B)^u_w}$$

There are two critical inference steps going on here. One is the highlighted $0L$, which relies on the fact that the interpretation of 0 is uniformly always false, at any pair of worlds, so the 'mismatch' between the left and right side of the sequent doesn't matter. The other is $\forall L$, which choose an instantiation for the variable that somehow *doesn't make sense* given the intended meaning of the intervals involved. Namely, it sets $x = w$ which is outside the interval $[t, v]$.

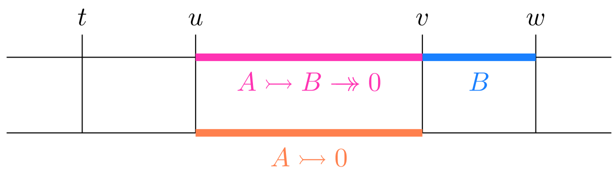
Here's a diagrammatic story of the proof: We start with the goal



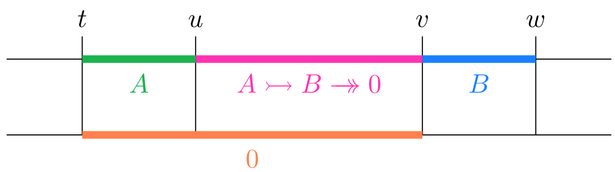
Assumptions are on the top line, conclusions are on the bottom line. We apply $\bullet R$ to split the context. We choose to split at v, so we must prove



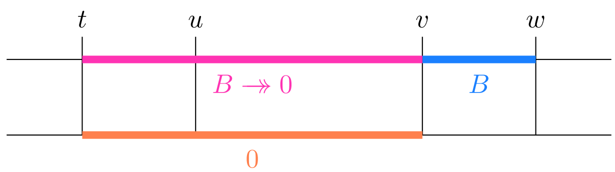
and



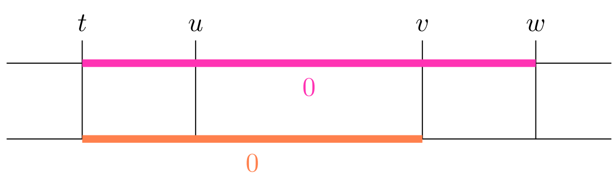
The former follows from the identity rule, so we concentrate on the latter. We apply $\rightarrowtail L$, extending the context with an extra interval:



We apply $\rightarrowtail L$, which consumes the A on the left, leaving us with the goal



The semantics of $B \twoheadrightarrow 0$ expects us to choose a right-extention of the interval it is at such that the extention satisfies B, and guarantees us that the whole extended interval will satsify 0 — but at this stage it is apparent in what sense choosing w is somehow 'wrong' — it's *outside* the interval of the conclusion! Although we can proceed to



and complete the proof, it feels like cheating. The assumption B was supposed to be 'consumed' by the earlier use of $\bullet R$.

The solution to this problem will be a form of *tethering* assumptions to a notion of *currently active* interval.

3 Tethering

The idea of tethering [Pfe10] comes out of trying to explain the way in which Pfenning-Davies modal logic [PD01] is a constructive S4 logic of possible worlds.

There is a similar counterexample that shows that it's incomplete to suppose that one can simply interpret $\Box$ as meaning 'for all future worlds' in the sense of the quantifier $\forall$ in constructive first-order logic, without any further constraints. This interpretation is essentially what's given by Simpson [Sim94]. The fix is to refine the semantics by adding a notion of *tethering* assumptions to a *currently active* world. We'll imitate and generalize this below, to have a notion of tethering to a certain *interval* of kripke worlds, instead of a single one.

3.1 Modal Logic Without Tethering

Let's first recapitulate the untethered modal semantics here for easy comparison with the situation above. The language of modal logic is:

$$\begin{array}{rcl} \text{Propositions } A & ::= & p \mid \Box A \mid A \Rightarrow B \mid \\ & & 0 \mid A \oplus B \mid \top \mid A \,\&\, B \end{array}$$

We say that every modal logic proposition is to be interpreted as a unary relation on some Kripke-like syntactic sort of worlds, on which there is a reflexive, transitive relation $\leq$. We require for every atom p a base relation $\eta_p(u)$ on worlds be provided, and then we can define A_u, a function whose inputs are an ordered language proposition A and a world u, and which outputs a first-order proposition. It is defined by recursion on A:

$$(\Box A)_u = \forall v \geq u. A_v$$

$$(A \Rightarrow B)_u = A_u \Rightarrow B_u$$

$$0_u = \bot \qquad (A \oplus B)_u = A_u \vee B_u$$

$$\top_u = \top \qquad (A \,\&\, B)_u = A_u \wedge B_u$$

$$p_u = \eta_p(u)$$

Interpretation of contexts is conjunctive, treating comma like &.

We write $\vdash_{PD}$ provability in Pfenning-Davies modal logic. As before, we have soundness

Lemma 3.1 *If $\Gamma \vdash_{PD} A$, then $\Gamma_u \vdash A_u$.*

but not completeness. A counterexample is $\neg\neg\Box\neg A \vdash \Box\neg A$. This is interpreted as

$$\neg\neg\forall v \geq u. \neg A_v \vdash \forall v \geq u. \neg A_v$$

and we can prove it as

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A_v \vdash A_v}}{A_v, \neg A_v \vdash}}{v \geq u, A_v, \forall v \geq u. \neg A_v \vdash}}{v \geq u, A_v \vdash \neg\forall v \geq u. \neg A_v}}{v \geq u, \neg\neg\forall v \geq u. \neg A_v, A_v \vdash}}{\neg\neg\forall v \geq u. \neg A_v \vdash \forall v \geq u. \neg A_v}$$

In Pfenning-Davies, we cannot make progress because we have $\Box$ on the right (which would clear the context) and $\neg$ on the left (which would clear the right-hand side if used).

3.2 Modal Logic With Tethering

I described a tethered semantics for Pfenning-Davies modal logic in [Ree10]. The gist of it is that you when pass to a polarized language and focused proof theory, the shift operators either produce or consume a linear *token* which represents the sequent's notion of the *current* world. The context-clearing effect of the $\Box R$ rule is achieved by actually statefully updating the current world. Every assumption in the context, by virtue of how the shifts work, must pass a check that it is at the current world before being further decomposed.

The modal language when polarized becomes

$$\begin{array}{rcl} \text{Positive Props } P & ::= & p \mid \Box N \mid 0 \mid P \oplus P \mid {\downarrow} N \\ \text{Negative Props } N & ::= & n \mid P \Rightarrow N \mid \top \mid N \mathbin{\&} N \mid {\uparrow} P \end{array}$$

and the semantics becomes

$$({\downarrow} N)_u = \#_u \multimap N_u \qquad ({\uparrow} P)_u = \#_u \otimes P_u$$

$$(\Box N)_u = \forall v \geq u.(\#_v \multimap N_v)$$

$$(P \Rightarrow N)_u = P_u \Rightarrow N_u$$

$$0_u = \bot \qquad (P \oplus P)_u = P_u \vee P'_u$$

$$\top_u = \top \qquad (N \mathbin{\&} N')_u = N_u \wedge N'_u$$

$$p_u = \eta_p(u) \qquad n_u = \eta_n(u)$$

The counterexample $\neg\neg\Box\neg A \Rightarrow \Box\neg A$ is polarized (assuming A is positive) as

$${\downarrow}\neg{\downarrow}\neg\Box\neg A \vdash {\uparrow}\Box\neg A$$

and so it's translated by the semantics to (abbreviating $\#_u \multimap$ as U_u and $\#_u \otimes$ as F_u)

$$\#_u, U_u \neg U_u \neg \forall v \geq u.U_v \neg A_v \vdash F_u \forall v \geq u.U_v \neg A_v$$

And we find, correctly, that this cannot be proved. Either we focus on the right, spending the token $\#_u$, and getting back a token $\#_v$ that can't be used on the left, or else we focus on the left, spending the token and obtaining only the unprovable goal

$$\vdash U_u \neg \forall v \geq u.U_v \neg A_v$$

which has clobbered the original right-hand side of the sequent.

4 Linear Interval Assumptions

Our goal is now to describe the data to which we will be tethered in the ordered case. Recall that the mental image we're supposed to have in ordinary linear logic is that the context is a soup of propositions that behave like resources, in that they cannot be freely duplicated or erased, although they have no fixed order. We mean to augment this picture with one special family of linear atomic propositions $\langle u, v\rangle$ which are supposed to mean 'a resource is available everywhere in the interval between u and v'. Such an assumption is not contractible or weakenable; having one resource between u and v cannot be turned into having zero or two resources there.

However, we will postulate that worlds come equipped with a transitive total order $<$, and that a context consisting of two adjoining interval-resource-assumptions should be considered definitionally equal to a context with the concatenated interval:

$$\frac{\Delta \vdash u < v \qquad \Delta \vdash v < w}{\Delta, \langle u, v\rangle, \langle v, w\rangle = \Delta, \langle u, w\rangle} \tag{$\dagger$}$$

This requirement would be satisfied if $\langle u, v\rangle$ were thought of as either orientation of half-open interval of, e.g. real numbers, $(u, v]$ or $[u, v)$. We side-step the question of whether endpoints are thought of as included on one end or the other by only depending on the merging postulate ($\dagger$).

5 Ordered Logic via Tethering

Now we say what the semantics is.

The polarized language of ordered logic is

$$\begin{array}{lcl} \text{Positive Props } P & ::= & p \mid 1 \mid P \bullet P \mid 0 \mid P \oplus P \mid {\downarrow}N \\ \text{Negative Props } N & ::= & n \mid P \twoheadrightarrow N \mid P \rightarrowtail N \mid \top \mid N \mathbin{\&} N \mid {\uparrow}P \end{array}$$

and the semantics of propositions is (abbreviating $x \in (u, v) = u < x \wedge x < v$)

$$({\uparrow}P)^u_v = \langle u, v\rangle \otimes P^u_v \qquad ({\downarrow}N)^u_v = \langle u, v\rangle \multimap N^u_v$$

$$(P \twoheadrightarrow N)^u_v = \forall x > v.\langle v, x\rangle \multimap P^v_x \Rightarrow N^u_x$$

$$(P \rightarrowtail N)^u_v = \forall x < u.\langle x, u\rangle \multimap P^x_u \Rightarrow N^x_v$$

$$1^u_v = (u = v) \qquad (P \bullet P')^u_v = \exists x \in (u, v).P^u_x \wedge (P')^x_v$$

$$0^u_v = \bot \qquad (P \oplus P')^u_v = P^u_v \vee (P')^u_v$$

$$\top^u_v = \top \qquad (N \mathbin{\&} N')^u_v = N^u_v \wedge (N')^u_v$$

$$p^u_v = \eta_p(u, v) \qquad n^u_v = \eta_n(u, v)$$

The semantics of contexts is the same as before,

$$\Omega^{u_0}_{u_n} = \exists u_1 \cdots u_{n-1}.(A_1)^{u_0}_{u_1} \wedge \cdots \wedge (A_n)^{u_{n-1}}_{u_n}$$

and the meaning of a sequent now includes a linear interval token: $\Omega \vdash A$ is translated as

$$\langle u, v\rangle, \Omega_v^u \vdash A_v^u$$

5.1 Visual Intuition for the Correctness of the Semantics

Before we get to claiming soundness and completeness, we want to provide some depictions of how the semantic clauses achieve the same effects as ordered logic.

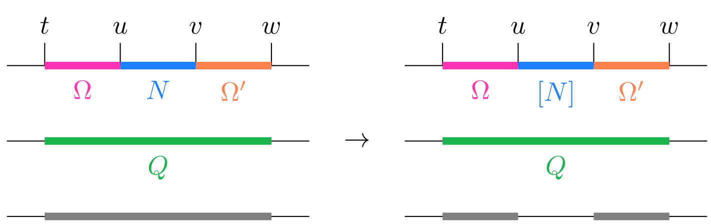
The behavior of $\downarrow$ on the left is to consume the interval of the shifted assumption. This corresponds to left focus in focused ordered logic

$$\frac{\Omega[N]\Omega' \vdash Q}{\Omega, N, \Omega' \vdash Q}$$

The semantics turns this into the partial derivation

$$\cfrac{\cfrac{\langle t, u\rangle, \langle v, w\rangle, \Omega_u^t[N_v^u](\Omega')_w^v \vdash Q_w^t \qquad \langle u, v\rangle \vdash \langle u, v\rangle}{\langle t, w\rangle, \Omega_u^t[\langle u, v\rangle \multimap N_v^u](\Omega')_w^v \vdash Q_w^t}}{\langle t, w\rangle, \Omega_u^t, (\uparrow N)_v^u, (\Omega')_w^v \vdash Q_w^t}$$

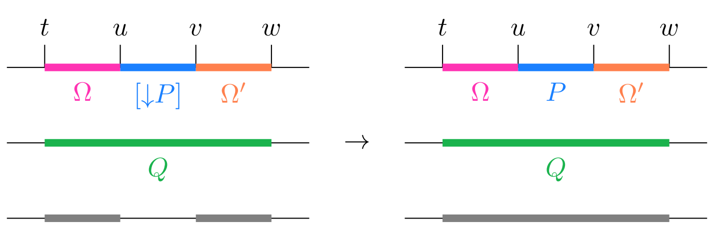
Visually, we draw the 'tether interval' as a gray bar. As we proceed in proof search *from* the bottom of the inference rule *toward* the top, the effect is the consumption of a portion of the tether.



The blur of a proposition on the left returns the tether:

$$\frac{\Omega, P, \Omega' \vdash Q}{\Omega[\downarrow P]\Omega' \vdash Q}$$

corresponds to



Observe that crucially a proposition on the left whose interval is *not entirely contained* by the tether interval *cannot* be focused on.

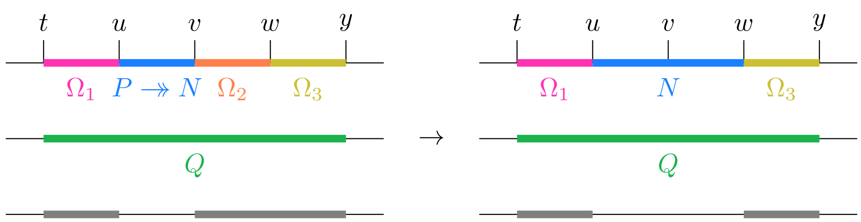
On the right, the entire tether is consumed by right focus, and is put back by right-blur.

We now consider how $\twoheadrightarrow$ behaves on the left. The inference rule in ordered logic is

$$\frac{\Omega_2 \vdash [P] \qquad \Omega_1[N]\Omega_3 \vdash Q}{\Omega_1[P \twoheadrightarrow N]\Omega_2, \Omega_3 \vdash Q}$$

The semantic translation of this ends up being the derivation

$$\frac{\frac{\frac{\frac{(\Omega_2)^v_w \vdash [P^v_w] \qquad \langle t,u\rangle, \langle w,y\rangle, (\Omega_1)^t_u[N^u_w](\Omega_3)^w_y \vdash Q^t_y}{\langle t,u\rangle, \langle w,y\rangle, (\Omega_1)^t_u[P^v_w \Rightarrow N^u_w](\Omega_2)^v_w(\Omega_3)^w_y \vdash Q^t_y}}{\langle t,u\rangle, \langle v,y\rangle, (\Omega_1)^t_u[\langle w,x\rangle \multimap P^v_w \Rightarrow N^u_w](\Omega_2)^v_w(\Omega_3)^w_y \vdash Q^t_y}}{\langle t,u\rangle, \langle v,y\rangle, (\Omega_1)^t_u[\forall x > v.\langle v,x\rangle \multimap P^v_x \Rightarrow N^u_x](\Omega_2)^v_w(\Omega_3)^w_y \vdash Q^t_y}}{\langle t,u\rangle, \langle v,y\rangle, (\Omega_1)^t_u[(P \twoheadrightarrow N)^u_v](\Omega_2)^v_w(\Omega_3)^w_y \vdash Q^t_y}$$



5.2 Correctness

We come now to stating the precise sense in which semantics is correct:

Theorem 5.1 (Soundness)

1. *If* $\Omega \vdash Q$, *then* $\langle u, v\rangle, \Omega^u_v \vdash Q^u_v$.
2. *If* $\Omega \vdash [P]$, *then* $\Omega^u_v \vdash [P^u_v]$.
3. *If* $\Omega[N]\Omega' \vdash Q$, *then* $\Omega^t_u[N^u_v](\Omega')^v_w \vdash Q^t_w$.

Proof By induction on the ordered logic derivation. ■

For completeness, we need a relation between ordered logic contexts Ω and (unrestricted) contexts Γ in the target language of the semantics, written $\Gamma \sim^u_v \Omega$, thought of as meaning 'a way to understand what Γ assumes about the interval $\langle u, v\rangle$ is as the ordered context Ω', or, more concisely, 'Γ can be read as Ω'. The general pattern of the completeness theorem will be statements of the form 'if something can be proved in the target language, then there is some reading of the context there that can be proved in ordered logic'.

We therefore define $\Gamma \sim^{u_0}_{u_n} \Omega$ when $\Omega = N_1, \ldots, N_n$ to mean that there are worlds $u_1, \ldots, u_{n-1}$ such that $\Gamma \vdash u_0 < \cdots < u_n$ and for every $i \in 1 \ldots n$, we have $(N_i)^{u_{i-1}}_{u_i} \in \Gamma$.

An easy lemma to prove is that we can concatenate readings in a suitable sense:

Lemma 5.2 *If* $\Gamma \sim^u_v \Omega$ *and* $\Gamma \sim^v_w \Omega'$, *then* $\Gamma \sim^u_w \Omega, \Omega'$.

We now state completeness:

Theorem 5.3 (Completeness)

1. *If* $\Gamma \vdash [P^u_v]$, *then there exists an* Ω *with* $\Gamma \sim^u_v \Omega$ *and* $\Omega \vdash [P]$.
2. *If* $\langle t,u\rangle, \langle v,w\rangle, \Gamma[N^u_v] \vdash Q^t_w$ *then at least one of the following is true:*
   (a) *there exist* Ω_1 *and* Ω_2 *such that*
   $$\Gamma \sim^t_u \Omega_1 \qquad \Gamma \sim^v_w \Omega_2 \qquad \Omega_1[N]\Omega_2 \vdash Q$$
   (b) *there exists* Ω *such that* $\Gamma \sim^t_w \Omega$ *and* $\Omega \vdash Q$
3. *If* $\Gamma, \langle t,w\rangle \vdash Q^t_w$, *then there exists* Ω *such that* $\Gamma \sim^t_w \Omega$ *and* $\Omega \vdash Q$.

Proof We give some representative cases.

Case: $\twoheadrightarrow$

We have a derivation that has the form

$$\dfrac{\dfrac{\Gamma \vdash [P^v_y] \qquad \langle t,u\rangle, \langle y,w\rangle, \Gamma[N^u_y] \vdash Q^t_w}{\langle t,u\rangle, \langle y,w\rangle, \Gamma[P^v_y \to N^u_y] \vdash Q^t_w} \qquad y \in (v,w)}{\langle t,u\rangle, \langle v,w\rangle, \Gamma[\forall x > v. \langle v,x\rangle \multimap P^v_x \to N^u_x] \vdash Q^t_w}$$

The reason that whatever y that was chosen as the instantiation of the $\forall L$-quantified x must be in the interval (v,w) comes half from the explicit requirement that $x > v$ in the semantics of $\twoheadrightarrow$, and half from the fact that the tether-interval consumption $\langle v,x\rangle \multimap \cdots$ *can't succeed* if $x > w$, because only the interval $\langle v,w\rangle$ is available. This phenomenon is what directly rules out the kind of strange 'reuse' of previously consumed hypotheses as in §2.1 simply because they still happen to be lying around in the translated context Γ.

We appeal to the induction hypothesis on P and N, splitting cases on what comes back from N. In case (a) for N, we get $\Omega_1, \Omega_2, \Omega_3$ such that

$$\Gamma \sim^v_y \Omega_2 \qquad \Gamma \sim^t_u \Omega_1 \qquad \Gamma \sim^y_w \Omega_3$$

and

$$\Omega_2 \vdash [P] \qquad \Omega_1[N]\Omega_3 \vdash Q$$

so it's straightforward to use the above lemma to conclude $\Gamma \sim^v_w \Omega_2, \Omega_3$ and derive

$$\dfrac{\Omega_2 \vdash [P] \qquad \Omega_1, [N], \Omega_3 \vdash Q}{\Omega_1[P \twoheadrightarrow N]\Omega_2\Omega_3 \vdash Q}$$

In case (b) for N, we immediately return the same reading.

Case: $\uparrow P$

We have a derivation of the form

$$\dfrac{\dfrac{\langle t, w\rangle, \Gamma, P^u_v \vdash Q^t_w}{\langle t, u\rangle, \langle v, w\rangle, \Gamma[\uparrow(\langle u, v\rangle \otimes P^u_v)] \vdash Q^t_w}}{\langle t, u\rangle, \langle v, w\rangle, \Gamma[(\uparrow P)^u_v] \vdash Q^t_w}$$

By i.h., there exists some Ω and some reading $(\Gamma, (\uparrow P)^u_v) \sim^t_w \Omega$. This reading either uses P^u_v or it doesn't. If it does, it breaks up into two smaller readings (neither of which uses P, because otherwise we would have a cycle in $<$) which we can call $\Gamma \sim^t_u \Omega_1$ and $\Gamma \sim^v_w \Omega_2$. Therefore we have $\Omega_1, P, \Omega_2 \vdash Q$. From that we can derive $\Omega_1[\uparrow P]\Omega_2 \vdash Q$ as required. Otherwise, the reading obtained from the i.h. doesn't use P^u_v, and so we just know we have a reading $\Gamma \sim^t_w \Omega$ and $\Omega \vdash Q$, which also suffices for the conclusion. ■

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