LF with Irrelevance

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December 9, 2001

Contents

1 Language

Kinds	K	$::= \mathsf{type} \mid \Pi x{:}A.K \mid \Pi x{\div}A.K$
Types	A	$::= a\ M \mid a \circ M \mid \Pi x{:}A.B \mid \Pi x{\div}A.B$
Objects	M	$::= x \mid c \mid \lambda x{:}A.M \mid \lambda x{\div}A.M \mid M_1\ M_2 \mid M_1 \circ M_2$
Signatures	Σ	$::= \cdot \mid \Sigma, a : K \mid \Sigma, c : A$
Contexts	Γ	$::= \cdot \mid \Gamma, x : A \mid \Gamma, x \div A$

2 Contexts

Judgments:

Valid Signatures	$\vdash \Sigma\ \mathsf{sig}$	Σ is a valid signature.
Valid Contexts	$\vdash \Gamma\ \mathsf{ctx}$	Γ is a valid context.
Context Equality	$\vdash \Gamma = \Gamma' : \mathsf{ctx}$	Γ and Γ' are equal contexts.
Context Ordering	$\vdash \Gamma \preceq \Gamma' : \mathsf{ctx}$	Γ is weaker than Γ'.

2.1 Valid Signatures

$$\frac{}{\vdash \cdot\ \mathsf{sig}}\ \mathsf{vs\_empty}$$

$$\frac{\vdash \Sigma\ \mathsf{sig} \qquad \cdot \vdash_\Sigma A : \mathsf{type}}{\vdash \Sigma, c : A\ \mathsf{sig}}\ \mathsf{vs\_cons}$$

$$\frac{\vdash \Sigma\ \mathsf{sig} \qquad \cdot \vdash_\Sigma K : \mathsf{kind}}{\vdash \Sigma, a : K\ \mathsf{sig}}\ \mathsf{vs\_fam}$$

All turnstiles in the sequel are implicitly paramaterized by a signature Σ.

2.2 Valid Contexts

$$\frac{}{\vdash \cdot\ \mathsf{ctx}}\ \mathsf{vc\_empty}$$

$$\frac{\vdash \Gamma : \mathsf{ctx} \qquad \Gamma \vdash A : \mathsf{type}}{\vdash \Gamma, x \star A\ \mathsf{ctx}}\ \mathsf{vc\_var}$$

2.3 Context Equality

$$\frac{}{\vdash \cdot = \cdot : \mathsf{ctx}}\ \mathsf{ce\_nil}$$

$$\frac{\vdash \Gamma = \Gamma' : \mathsf{ctx} \qquad \Gamma' \vdash A = A' : \mathsf{type}}{\vdash \Gamma, x \star A = \Gamma', x \star A' : \mathsf{ctx}}\ \mathsf{ce\_var}$$

2.4 Context Ordering

$$\frac{}{\cdot \preceq \cdot}\ \mathsf{co\_nil}$$

$$\frac{\Gamma \preceq \Gamma'}{\Gamma, x \star A \preceq \Gamma', x \star A}\ \mathsf{co\_eq}$$

$$\frac{\Gamma \preceq \Gamma'}{\Gamma, x \div A \preceq \Gamma', x : A}\ \mathsf{co\_lt}$$

3 Proof Typing

Functions:

Proof Promotion	$\Gamma \mapsto \Gamma^{\oplus}$	Promote proof variables in Γ to term variables

Judgments:

Proof Object Typing	$\Gamma \vdash M \div A$	In Γ, M has irrelevant type A.

3.1 Proof Promotion

$$\begin{array}{rcl} .^{\oplus} & := & \cdot \\ (\Gamma, x : A)^{\oplus} & := & \Gamma^{\oplus}, x : A \\ (\Gamma, x \div A)^{\oplus} & := & \Gamma^{\oplus}, x : A \end{array}$$

3.2 Proof Object Typing

$$\frac{\Gamma^{\oplus} \vdash M : A \qquad \Gamma \vdash A : \mathsf{type}}{\Gamma \vdash M \div A}\ \mathsf{pot}$$

4 Typing

Judgments:

Object Typing	$\Gamma \vdash M : A$	In Γ, M has type A.
Type Validity	$\Gamma \vdash A : \mathsf{type}$	In Γ, A is a valid type.
Kind Validity	$\Gamma \vdash K : \mathsf{kind}$	In Γ, K is a valid kind.

4.1 Object Typing

$$\frac{c : A \in \Sigma}{\Gamma \vdash c : A}\ \mathsf{ot\_const}$$

$$\frac{x : A \in \Gamma}{\Gamma \vdash x : A}\ \mathsf{ot\_var}$$

$$\frac{\Gamma \vdash A : \mathsf{type} \qquad \Gamma, x \star A \vdash M : B}{\Gamma \vdash \lambda x{\star}A.M : \Pi x{\star}A.B}\ \mathsf{ot\_lam}$$

$$\frac{\Gamma \vdash M_1 : \Pi x{\star}A.B \qquad \Gamma \vdash M_2 \star A}{\Gamma \vdash M_1 * M_2 : [M_2/x]B}\ \mathsf{ot\_app}$$

$$\frac{\Gamma \vdash M : A \qquad \Gamma \vdash A = B : \mathsf{type}}{\Gamma \vdash M : B}\ \mathsf{ot\_conv}$$

4.2 Type Validity

$$\frac{a : K \in \Sigma}{\Gamma \vdash a : K}\ \mathsf{tv\_const}$$

$$\frac{\Gamma \vdash A : \mathsf{type} \qquad \Gamma, x \star A \vdash B : \mathsf{type}}{\Gamma \vdash \Pi x{\star}A.B : \mathsf{type}}\ \mathsf{tv\_pi}$$

$$\frac{\Gamma \vdash A_1 : \Pi x{\star}A_2.K \qquad \Gamma \vdash M \star A_2}{\Gamma \vdash A_1 * M : [M/x]K}\ \mathsf{tv\_app}$$

$$\frac{\Gamma \vdash A : K \qquad \Gamma \vdash K = L : \mathsf{type}}{\Gamma \vdash A : L}\ \mathsf{tv\_conv}$$

4.3 Kind Validity

$$\frac{}{\Gamma \vdash \mathsf{type} : \mathsf{kind}}\ \mathsf{kv\_type}$$

$$\frac{\Gamma \vdash A : \mathsf{type} \qquad \Gamma, x \star A \vdash K : \mathsf{kind}}{\Gamma \vdash \Pi x{\star}A.K : \mathsf{kind}}\ \mathsf{kv\_pi}$$

5 Equality

Judgments:

Proof Equality	$\Gamma \vdash M_1 = M_2 \div A$	In Γ, $M_1 = M_2$ at irrelevant type A.
Object Equality	$\Gamma \vdash M_1 = M_2 : A$	In Γ, $M_1 = M_2$ at type A.
Type Equality	$\Gamma \vdash A_1 = A_2 : \mathsf{type}$	In Γ, A_1 and A_2 are equal types.
Kind Equality	$\Gamma \vdash K_1 = K_2 : \mathsf{kind}$	In Γ, K_1 and K_2 are equal kinds.

5.1 Proof Equality

$$\frac{\Gamma^{\oplus} \vdash M = M : A \qquad \Gamma^{\oplus} \vdash N = N : A \qquad \Gamma \vdash A : \mathsf{type}}{\Gamma \vdash M = N \div A}\ \mathsf{pe}$$

5.2 Object Equality

5.2.1 Simultaneous Congruence

$$\frac{c : A \in \Sigma}{\Gamma \vdash c = c : A}\ \mathsf{oe\_const}$$

$$\frac{x : A \in \Gamma}{\Gamma \vdash x = x : A}\ \mathsf{oe\_var}$$

$$\frac{\Gamma \vdash A = A' : \mathsf{type} \qquad \Gamma \vdash A = A'' : \mathsf{type} \qquad \Gamma, x \star A \vdash M = M' : B}{\Gamma \vdash \lambda x{\star}A'.M = \lambda x{\star}A''.M' : \Pi x{\star}A.B}\ \mathsf{oe\_lam}$$

$$\frac{\Gamma \vdash M_1 = M_1' : \Pi x{\star}A.B \qquad \Gamma \vdash M_2 = M_2' \star A}{\Gamma \vdash M_1 * M_2 = M_1' * M_2' : [M_2/x]B}\ \mathsf{oe\_app}$$

5.2.2 Extensionality

$$\frac{\Gamma \vdash A : \mathsf{type} \qquad \{\Gamma \vdash M, N : \Pi x{\star}A.B\} \qquad \Gamma, x \star A \vdash M * x = N * x : B}{\Gamma \vdash M = N : \Pi x{\star}A.B}\ \mathsf{oe\_ext}$$

5.2.3 Parallel Reduction

$$\frac{\{\Gamma \vdash A : \mathsf{type}\} \qquad \Gamma, x \star A \vdash M_2 = M_2' : B \qquad \Gamma \vdash M_1 = M_1' \star A}{\Gamma \vdash (\lambda x{\star}A.M_2) * M_1 = [M_1'/x]M_2' : [M_1/x]B}\ \mathsf{oe\_red}$$

5.2.4 Type Conversion

$$\frac{\Gamma \vdash M = N : A \qquad \Gamma \vdash A = B : \mathsf{type}}{\Gamma \vdash M = N : B}\ \mathsf{oe\_conv}$$

5.2.5 Equivalence

$$\frac{\Gamma \vdash M = N : A \qquad \Gamma \vdash N = O : A}{\Gamma \vdash M = O : A}\ \textsf{oe\_trans}$$

$$\frac{\Gamma \vdash M = N : A}{\Gamma \vdash N = M : A}\ \textsf{oe\_sym}$$

5.3 Type Equality

5.3.1 Simultaneous Congruence

$$\frac{a : K \in \Sigma}{\Gamma \vdash a = a : K}\ \textsf{te\_const}$$

$$\frac{\Gamma \vdash A = A' : \textsf{type} \qquad \{\Gamma \vdash A : \textsf{type}\} \qquad \Gamma, x \star A \vdash B = B' : \textsf{type}}{\Gamma \vdash \Pi x{\star}A.B = \Pi x{\star}A'.B' : \textsf{type}}\ \textsf{te\_pi}$$

$$\frac{\Gamma \vdash A = A' : \Pi x{\star}A.K \qquad \Gamma \vdash M = M' \star A}{\Gamma \vdash A * M = A' * M' : [M/x]K}\ \textsf{te\_app}$$

5.3.2 Kind Conversion

$$\frac{\Gamma \vdash A = B : K \qquad \Gamma \vdash K = L : \textsf{kind}}{\Gamma \vdash A = B : L}\ \textsf{te\_app}$$

5.3.3 Equivalence

$$\frac{\Gamma \vdash A = B : K \qquad \Gamma \vdash B = C : K}{\Gamma \vdash A = C : K}\ \textsf{te\_trans}$$

$$\frac{\Gamma \vdash A = B : K}{\Gamma \vdash B = A : K}\ \textsf{te\_sym}$$

5.4 Kind Equality

5.4.1 Simultaneous Congruence

$$\frac{}{\Gamma \vdash \textsf{type} = \textsf{type} : \textsf{kind}}\ \textsf{ke\_type}$$

$$\frac{\Gamma \vdash A = A' : \textsf{type} \qquad \{\Gamma \vdash A : \textsf{type}\} \qquad \Gamma, x \star A \vdash K = K' : \textsf{kind}}{\Gamma \vdash \Pi x{\star}A.K = \Pi x{\star}A'.K' : \textsf{kind}}\ \textsf{ke\_pi}$$

5.4.2 Equivalence

$$\frac{\Gamma \vdash K = L : \textsf{kind} \qquad \Gamma \vdash L = L' : \textsf{kind}}{\Gamma \vdash K = L' : \textsf{kind}}\ \textsf{ke\_trans}$$

$$\frac{\Gamma \vdash K = L : \textsf{kind}}{\Gamma \vdash L = K : \textsf{kind}}\ \textsf{ke\_sym}$$

6 Structural Lemmas

Lemma 6.1 (Weakening) *If* $\Gamma, \Gamma' \vdash J$, *then* $\Gamma, x \star A, \Gamma \vdash J$.

Proof Straightforward induction. ■

Lemma 6.2 (Reflexivity)

1. *If* $\Gamma \vdash M \star A$, *then* $\Gamma \vdash M = M \star A$.
2. *If* $\Gamma \vdash A : K$, *then* $\Gamma \vdash A = A : K$.
3. *If* $\Gamma \vdash K : \mathsf{kind}$, *then* $\Gamma \vdash K = K : \mathsf{kind}$.
4. *If* $\vdash \Gamma : \mathsf{ctx}$, *then* $\vdash \Gamma = \Gamma : \mathsf{ctx}$.

Proof Straightforward induction. ■

Lemma 6.3 (Substitution) *Suppose* $\vdash \Gamma, x \star A, \Gamma' : \mathsf{ctx}$. *If* $\Gamma \vdash M \star A$ *and* $\Gamma, x \star A, \Gamma' \vdash J$, *then* $\Gamma, [M/x]\Gamma' \vdash [M/x]J$.

Proof Straightforward induction. ■

Lemma 6.4 (Context Promotion) *If* $\Gamma \vdash J$ *and* $\Gamma \preceq \Gamma'$, *then* $\Gamma' \vdash J$.

Proof Straightforward induction on $\mathcal{D} :: \Gamma \vdash J$. Some representative cases:

Case:

$$\mathcal{D} = \dfrac{\overset{\mathcal{D}_1}{\Gamma \vdash A : \mathsf{type}} \quad \overset{\mathcal{D}_2}{\Gamma, x \star A \vdash M : B}}{\Gamma \vdash \lambda x{\star}A.M : \Pi x{\star}A.B}\ \mathsf{ot\_lam}$$

We apply the induction hypothesis to $\mathcal{D}_1$ to get that $\Gamma' \vdash A : \mathsf{type}$, and to $\mathcal{D}_2$ and the fact that $\Gamma, x \star A \preceq \Gamma', x \star A$ to obtain $\Gamma', x \star A \vdash M : B$, from which we conclude using $\mathsf{ot\_lam}$ that $\Gamma' \vdash \lambda x{\star}A.M : \Pi x{\star}A.B$.

Case:

$$\mathcal{D} = \dfrac{\overset{\mathcal{D}_1}{\Gamma^{\oplus} \vdash M : A} \quad \overset{\mathcal{D}_2}{\Gamma \vdash A : \mathsf{type}}}{\Gamma \vdash M \div A}\ \mathsf{pot}$$

It is easily seen that $\Gamma'^{\oplus}$ is identical to $\Gamma^{\oplus}$, so that *a fortiori* $\Gamma^{\oplus} \preceq \Gamma'^{\oplus}$. Moreoever $\mathcal{D}_1$ is also a derivation of $\Gamma'^{\oplus} \vdash M : A$. We know, therefore, by the induction hypothesis applied to $\mathcal{D}_1$ and $\mathcal{D}_2$ that $\Gamma'^{\oplus} \vdash M : A$ and $\Gamma' \vdash A : \mathsf{type}$, so by pot we have $\Gamma' \vdash M \div A$.

■

Corollary 6.5 *If* $\Gamma \vdash J$, *then* $\Gamma^{\oplus} \vdash J$.

Lemma 6.6 (Context Equality Promotion) *If* $\vdash \Gamma_1 = \Gamma_2 : \mathsf{ctx}$*, then we have* $\vdash \Gamma_1^{\oplus} = \Gamma_2^{\oplus} : \mathsf{ctx}$.

Proof By induction on $\mathcal{D} :: (\vdash \Gamma_1 = \Gamma_2 : \mathsf{ctx})$.

Case:

$$\frac{}{\vdash \cdot = \cdot : \mathsf{ctx}}\,\mathsf{ce\_nil}$$

Immediately we have $\vdash \cdot^{\oplus} = \cdot^{\oplus} : \mathsf{ctx}$.

Case:

$$\frac{\overset{\mathcal{D}_1}{\vdash \Gamma_1 = \Gamma_2 : \mathsf{ctx}} \quad \overset{\mathcal{D}_2}{\Gamma_2 \vdash A = A' : \mathsf{type}}}{\vdash \Gamma_1, x \star A = \Gamma_2, x \star A' : \mathsf{ctx}}\,\mathsf{ce\_var}$$

By the induction hypothesis on $\mathcal{D}_1$ we know $\vdash \Gamma_1^{\oplus} = \Gamma_2^{\oplus} : \mathsf{ctx}$. Applying Corollary 6.5 to $\mathcal{D}_2$ provides $\Gamma_2^{\oplus} \vdash A = A' : \mathsf{type}$. So we can apply $\mathsf{ce\_var}$ to obtain $\vdash \Gamma_1^{\oplus}, x : A = \Gamma_2^{\oplus}, x : A$.

■

Lemma 6.7 (Context Conversion) *Suppose* $\vdash \Gamma = \Gamma' : \mathsf{ctx}$*. If* $\Gamma \vdash J$*, then* $\Gamma' \vdash J$.

Proof By induction on $\mathcal{D} :: \Gamma \vdash J$. Some important cases:

Case:

$$\frac{\overset{\mathcal{D}_1}{\Gamma^{\oplus} \vdash M : A} \quad \overset{\mathcal{D}_2}{\Gamma \vdash A : \mathsf{type}}}{\Gamma \vdash M \div A}\,\mathsf{pot}$$

By Lemma 6.6, $\vdash \Gamma^{\oplus} = \Gamma'^{\oplus} : \mathsf{ctx}$ so the induction hypothesis on $\mathcal{D}_1, \mathcal{D}_2$ gives $\Gamma'^{\oplus} \vdash M : A$ and $\Gamma' \vdash A : \mathsf{type}$, hence $\Gamma' \vdash M \div A$.

Case:

$$\mathcal{D} = \frac{x : A \in \Gamma}{\Gamma \vdash x : A}\,\mathsf{ot\_var}$$

Since $\vdash \Gamma = \Gamma' : \mathsf{ctx}$, we know by inversion on the context validity rules that Γ' is of the form $\Gamma'_1, x : A', \Gamma'_2$ such that $\Gamma'_1 \vdash A = A' : \mathsf{type}$. By weakening, $\Gamma' \vdash A = A' : \mathsf{type}$. Therefore

$$\frac{\dfrac{x : A' \in \Gamma'}{\Gamma' \vdash x : A'}\,\mathsf{ot\_var} \quad \dfrac{\Gamma' \vdash A = A' : \mathsf{type}}{\Gamma' \vdash A' = A : \mathsf{type}}\,\mathsf{te\_sym}}{\Gamma' \vdash x : A}\,\mathsf{ot\_conv}$$

Case:

$$\mathcal{D} = \cfrac{\begin{array}{cc}\mathcal{D}_1 & \mathcal{D}_2 \\ \Gamma \vdash A : \mathsf{type} & \Gamma, x \star A \vdash M : B\end{array}}{\Gamma \vdash \lambda x{\star}A.M : \Pi x{\star}A.B}\,\mathsf{ot\_lam}$$

Applying the induction hypothesis to $\mathcal{D}_1$ gives $\Gamma' \vdash A : \mathsf{type}$. Reflexivity implies $\Gamma' \vdash A = A : \mathsf{type}$, hence $\vdash \Gamma, x \star A = \Gamma', x \star A$ using $\mathsf{ce\_var}$. Therefore we can apply the induction hypothesis to $\Gamma, x \star A \vdash M : B$ to obtain $\Gamma', x \star A \vdash M : B$, and conclude using $\mathsf{ot\_lam}$ that $\Gamma' \vdash \lambda x{\star}A.M : \Pi x{\star}A.B$.

Case: $\mathcal{D} =$

$$\cfrac{\begin{array}{ccc}\mathcal{D}_1 & \mathcal{D}_2 & \mathcal{D}_3 \\ \Gamma \vdash A = A' : \mathsf{type} & \Gamma \vdash A = A'' : \mathsf{type} & \Gamma, x \star A \vdash M = M' : B\end{array}}{\Gamma \vdash \lambda x{\star}A'.M = \lambda x{\star}A''.M' : \Pi x{\star}A.B}\,\mathsf{oe\_lam}$$

Applying the induction hypothesis to $\mathcal{D}_1$ provides a derivation $\mathcal{D}_1' :: \Gamma' \vdash A = A' : \mathsf{type}$, which we can use to infer

$$\cfrac{\vdash \Gamma = \Gamma' : \mathsf{ctx} \quad \cfrac{\begin{array}{c}\mathcal{D}_1' \\ \Gamma' \vdash A = A' : \mathsf{type}\end{array} \quad \cfrac{\begin{array}{c}\mathcal{D}_1' \\ \Gamma' \vdash A = A' : \mathsf{type}\end{array}}{\Gamma' \vdash A' = A : \mathsf{type}}\,\mathsf{te\_sym}}{\Gamma' \vdash A = A : \mathsf{type}}\,\mathsf{te\_trans}}{\vdash \Gamma, x \star A = \Gamma', x \star A : \mathsf{ctx}}\,\mathsf{ce\_var}$$

so that we can apply the induction hypothesis to $\mathcal{D}_3$ to obtain $\Gamma', x \star A \vdash M = M' : B$. Finally, we apply the induction hypothesis to $\mathcal{D}_2$ to obtain $\Gamma' \vdash A = A'' : \mathsf{type}$ and conclude that

$$\Gamma' \vdash \lambda x \star A'.M = \lambda x \star A''.M : \Pi x{\star}A.B$$

using $\mathsf{oe\_lam}$.

Case:

$$\mathcal{D} = \cfrac{\begin{array}{ccc}\mathcal{D}_1 & \mathcal{D}_2 & \mathcal{D}_3 \\ \Gamma^{\oplus} \vdash M = M : A & \Gamma^{\oplus} \vdash N = N : A & \Gamma \vdash A : \mathsf{type}\end{array}}{\Gamma \vdash M = N \div A}\,\mathsf{pe}$$

By Lemma 6.6, we know $\vdash \Gamma^{\oplus} = \Gamma'^{\oplus} : \mathsf{ctx}$, so the induction hypothesis on $\mathcal{D}_1, \mathcal{D}_2$ provides $\Gamma'^{\oplus} \vdash M = M : A$ and $\Gamma'^{\oplus} \vdash N = N : A$. The induction hypothesis applied to $\mathcal{D}_3$ gives $\Gamma' \vdash A : \mathsf{type}$, so pe implies $\Gamma' \vdash M = N \div A$. ■

Lemma 6.8 (Context Equality Symmetry) *If* $\vdash \Gamma = \Gamma' : \mathsf{ctx}$, *then* $\vdash \Gamma' = \Gamma : \mathsf{ctx}$.

Proof By induction on $\mathcal{D} :: (\vdash \Gamma = \Gamma' : \mathsf{ctx})$.

Case:

$$\mathcal{D} = \frac{}{\vdash \cdot = \cdot : \mathsf{ctx}} \mathsf{ce\_nil}$$

Immediate.

Case:

$$\frac{\overset{\mathcal{D}_1}{\vdash \Gamma = \Gamma' : \mathsf{ctx}} \quad \overset{\mathcal{D}_2}{\Gamma' \vdash A = A' : \mathsf{type}}}{\vdash \Gamma, x \star A = \Gamma', x \star A' : \mathsf{ctx}} \mathsf{ce\_var}$$

By the induction hypothesis applied to $\mathcal{D}_1$, we have $\vdash \Gamma' = \Gamma : \mathsf{ctx}$. By Lemma 6.7 this fact together with $\mathcal{D}_2$ implies that $\Gamma \vdash A = A' : \mathsf{type}$, and by $\mathsf{tv\_sym}$ we deduce $\Gamma \vdash A' = A : \mathsf{type}$, so by $\mathsf{ce\_var}$ we have $\vdash \Gamma', x \star A' = \Gamma, x \star A : \mathsf{ctx}$.

■

Lemma 6.9 (Functionality) *Suppose*

$$\mathcal{D} :: (\vdash \Gamma, y \star D, \Gamma' \; \mathsf{ctx})$$

$$\Gamma \vdash N \star D \qquad \Gamma \vdash N' \star D$$

$$\Gamma \vdash N = N' \star D$$

then

1. $\vdash \Gamma, [N/y]\Gamma' = \Gamma, [N'/y]\Gamma' : \mathsf{ctx}$.
2. *If* $\mathcal{E} :: \Gamma, y \star D, \Gamma' \vdash M : A$ *then* $\Gamma, [N/y]\Gamma' \vdash [N/y]P = [N'/y]M : [N/y]A$.
3. *If* $\mathcal{E} :: \Gamma, y \star D, \Gamma' \vdash A : K$ *then* $\Gamma, [N/y]\Gamma' \vdash [N/y]A = [N'/y]A : [N/y]K$.
4. *If* $\mathcal{E} :: \Gamma, y \star D, \Gamma' \vdash K : \mathsf{kind}$ *then* $\Gamma, [N/y]\Gamma' \vdash [N/y]K = [N'/y]K : \mathsf{kind}$.

Proof By lexicographic induction on

- The number of inference rules in $\mathcal{E}$ (if any) plus the number of inference rules in $\mathcal{D}$.
- The size of the subject expression in $\mathcal{E}$, if any. (i.e. M in part 2, A in part 3, K in part 4)

1. We proceed by cases on the structure of $\mathcal{D}$.

Case:

$$\mathcal{D} = \frac{\overset{\mathcal{D}_1}{\vdash \Gamma : \mathsf{ctx}} \quad \overset{\mathcal{D}_2}{\Gamma \vdash D : \mathsf{type}}}{\vdash \Gamma, y \star D : \mathsf{ctx}} \mathsf{vc\_var}$$

By reflexivity and $\mathcal{D}_1$ we have $\vdash \Gamma = \Gamma : \mathsf{ctx}$, as required.

Case:

$$\mathcal{D} = \dfrac{\overset{\mathcal{D}_1}{\vdash \Gamma, y \star D, \Gamma'' : \mathsf{ctx}} \quad \overset{\mathcal{D}_2}{\Gamma, y \star D, \Gamma'' \vdash A : \mathsf{type}}}{\vdash \Gamma, y \star D, \Gamma'', x \star' A\, \mathsf{ctx}}\ \mathsf{vc\_var}$$

The induction hypothesis (part 1) applied to $\mathcal{D}_1$ gives

$$\vdash \Gamma, [N/y]\Gamma'' = \Gamma, [N'/y]\Gamma'' : \mathsf{ctx}$$

The induction hypothesis (part 3) applied to $\mathcal{D}_2$ gives

$$\Gamma, [N/y]\Gamma'' \vdash [N/y]A = [N'/y]A : \mathsf{type}$$

Lemma 6.7 then implies that

$$\Gamma, [N'/y]\Gamma'' \vdash [N/y]A = [N'/y]A : \mathsf{type}$$

hence

$$\dfrac{\begin{array}{c}\vdash \Gamma, [N/y]\Gamma'' = \Gamma, [N'/y]\Gamma'' : \mathsf{ctx} \\ \Gamma, [N'/y]\Gamma'' \vdash [N/y]A = [N'/y]A : \mathsf{type}\end{array}}{\vdash \Gamma, [N/y]\Gamma'', x \star' [N/y]A = \Gamma, [N'/y]\Gamma'', x \star' [N'/y]A : \mathsf{ctx}}\ \mathsf{ce\_var}$$

2, 3, 4. We proceed by cases on the structure of $\mathcal{E}$. Some representative cases:

Case:

$$\mathcal{E} = \dfrac{x : A \in \Gamma, y \star D, \Gamma'}{\Gamma, y \star D, \Gamma' \vdash x : A}\ \mathsf{ot\_var}$$

If $x \neq y$, then we must show that $\Gamma, [N/y]\Gamma' \vdash x = x : [N/y]A$, which follows by substitution and reflexivity. If $x = y$, then we know that in fact A is identical to D and $\star$ is :, and we must show that $\Gamma, [N/y]\Gamma' \vdash N = N' : [N/y]D$. This is possible by first using weakening on the assumption $\Gamma \vdash N = N' : D$ to obtain $\Gamma, y : D, \Gamma' \vdash N = N' : D$, and then using substitution and the fact that y is fresh relative to N, N'. *[XXX: ?]*

Case:

$$\mathcal{E} = \dfrac{\overset{\mathcal{E}_1}{\Gamma, y \star D, \Gamma' \vdash M_1 : \Pi x {\div} A.B} \quad \overset{\mathcal{E}_2}{\Gamma, y \star D, \Gamma' \vdash M_2 \div A}}{\Gamma, y \star D, \Gamma' \vdash M_1 \circ M_2 : [M_2/x]B}\ \mathsf{ot\_app}$$

By inversion,

$$\mathcal{E}_2 = \dfrac{\overset{\mathcal{E}_2'}{\Gamma^{\oplus}, y : D, \Gamma'^{\oplus} \vdash M_2 : A} \quad \overset{\mathcal{E}_2''}{\Gamma, y \star D, \Gamma \vdash A : \mathsf{type}}}{\Gamma, y \star D, \Gamma' \vdash M_2 \div A}\ \mathsf{pot}$$

for some $\mathcal{E}_2', \mathcal{E}_2''$. By substitution of N, N' into $\mathcal{E}_2$ and reflexivity, we have

$$\Gamma, [N/y]\Gamma' \vdash [N/y]M_2 = [N/y]M_2 \div [N/y]A \qquad (\flat)$$

and

$$\Gamma, [N'/y]\Gamma' \vdash [N'/y]M_2 = [N'/y]M_2 \div [N'/y]A \qquad (*)$$

By the induction hypothesis (part 3) on $\mathcal{E}_2''$, we know

$$\Gamma, [N/y]\Gamma' \vdash [N/y]A = [N'/y]A : \mathsf{type} \qquad (**)$$

Also by the induction hypothesis (part 1) we know that

$$\vdash \Gamma, [N/y]\Gamma' = \Gamma, [N'/y]\Gamma' : \mathsf{ctx}$$

so by Lemma 6.8 we have

$$\vdash \Gamma, [N'/y]\Gamma' = \Gamma, [N/y]\Gamma' : \mathsf{ctx}$$

hence we can use Lemma 6.7 to infer from $(*)$ that

$$\Gamma, [N/y]\Gamma' \vdash [N'/y]M_2 = [N'/y]M_2 \div [N'/y]A$$

and by inversion obtain a derivation of

$$(\Gamma, [N/y]\Gamma')^{\oplus} \vdash [N'/y]M_2 = [N'/y]M_2 : [N'/y]A$$

Corollary 6.5 and $(**)$ together imply that

$$(\Gamma, [N/y]\Gamma')^{\oplus} \vdash [N/y]A = [N'/y]A : \mathsf{type}$$

so we can use $\mathsf{ot\_sym}$, $\mathsf{ot\_conv}$ to conclude

$$(\Gamma, [N/y]\Gamma')^{\oplus} \vdash [N'/y]M_2 = [N'/y]M_2 : [N/y]A \qquad (\sharp_2)$$

Now by inversion on $(\flat)$ we have a derivation

$$(\Gamma, [N/y]\Gamma')^{\oplus} \vdash [N/y]M_2 = [N/y]M_2 : [N/y]A \qquad (\sharp_1)$$

and substitution on $\mathcal{E}_2''$ provides

$$\Gamma, [N/y]\Gamma' \vdash [N/y]A : \mathsf{type} \qquad (\sharp_3)$$

so we use pe on $(\sharp_1)$, $(\sharp_2)$, and $(\sharp_3)$ to obtain

$$\Gamma, [N/y]\Gamma' \vdash [N/y]M_2 = [N'/y]M_2 \div [N/y]A$$

Finally, we can apply the induction hypothesis directly to $\mathcal{E}_1$ to see

$$\Gamma, [N/y]\Gamma' \vdash [N/y]M_1 = [N'/y]M_1 : [N/y]\Pi x{\div}A.B$$

and conclude using $\mathsf{oe\_app}$ that

$$\Gamma, [N/y]\Gamma' \vdash [N/y](M_1 \circ M_2) = [N'/y](M_1 \circ M_2) : [N/y][M_2/x]B$$

Case:

$$\mathcal{E} = \dfrac{\overset{\mathcal{E}_1}{\Gamma, y \star D, \Gamma' \vdash A : \mathsf{type}} \quad \overset{\mathcal{E}_2}{\Gamma, y \star D, \Gamma', x \div A \vdash M : B}}{\Gamma, y \star D, \Gamma' \vdash \lambda x \div A.M : \Pi x \div A.B}\,\mathsf{ot\_lam}$$

The induction hypothesis applied to $\mathcal{E}_1$ gives $\Gamma, [N/y]\Gamma' \vdash [N/y]A = [N'/y]A : \mathsf{type}$. We can also apply the induction hypothesis to $\mathcal{E}_2$, though it is not obvious that we can, since the context is larger. However, we can prove $\Gamma, y \star D, \Gamma', x \div A : \mathsf{ctx}$ by

$$\mathcal{D}' = \dfrac{\overset{\mathcal{D}}{\Gamma, y \star D, \Gamma' : \mathsf{ctx}} \quad \overset{\mathcal{E}_1}{\Gamma, y \star D, \Gamma' \vdash A : \mathsf{type}}}{\vdash \Gamma, y \star D, \Gamma', x \div A : \mathsf{ctx}}\,\mathsf{vc\_var}$$

so that the number of inference rules in the two derivations $|\mathcal{D}'| + |\mathcal{E}_2| = (|\mathcal{D}| + |\mathcal{E}_1| + 1) + |\mathcal{E}_2| = |\mathcal{D}| + (|\mathcal{E}_1| + |\mathcal{E}_2| + 1) = |\mathcal{D}| + |\mathcal{E}|$ stays the same, but the subject expression is smaller — it changes from $\lambda x \div A.M$ to M. So we obtain

$$\Gamma, [N/y]\Gamma', x \div [N/y]A \vdash [N/y]M = [N'/y]M : [N/y]B$$

We then can use substitution of N for y in $\mathcal{E}_1$ and reflexivity to derive

$$\Gamma, [N/y]\Gamma' \vdash [N/y]A = [N/y]A : \mathsf{type}$$

and use $\mathsf{oe\_lam}$ to conclude

$$\Gamma, [N/y]\Gamma' \vdash [N/y](\lambda x : A.M) = [N'/y](\lambda x : A.M) : [N/y]B$$

■

7 Post-Functionality Lemmas

Lemma 7.1 (Inversion of Products)

1. *If* $\Gamma \vdash \Pi x{\star}A_1.A_2 : K$ *then* $\Gamma \vdash A_1 : \mathsf{type}$ *and* $\Gamma, x \star A_1 \vdash A_2 : \mathsf{type}$.
2. *If* $\Gamma \vdash \Pi x{\star}A.K : \mathsf{kind}$ *then* $\Gamma \vdash A : \mathsf{type}$ *and* $\Gamma, x \star A \vdash K : \mathsf{kind}$.

Proof By induction and inversion, respectively. ■

Lemma 7.2 (Validity)

1. *If* $\Gamma \vdash M \star A$, *then* $\Gamma \vdash A : \mathsf{type}$.
2. *If* $\Gamma \vdash M = N \star A$, *then* $\Gamma \vdash M \star A$, $\Gamma \vdash N \star A$, *and* $\Gamma \vdash A : \mathsf{type}$.
3. *If* $\Gamma \vdash A : K$, *then* $\Gamma \vdash K : \mathsf{kind}$.

4. *If* $\Gamma \vdash A = B : K$*, then* $\Gamma \vdash A : K$*,* $\Gamma \vdash B : K$*, and* $\Gamma \vdash K : \mathsf{kind}$.

5. *If* $\Gamma \vdash K = L : \mathsf{kind}$*, then* $\Gamma \vdash K : \mathsf{kind}$*, and* $\Gamma \vdash L : \mathsf{kind}$.

Proof Straightforward induction. Some representative cases:

Case:

$$\mathcal{D} = \frac{\overset{\mathcal{D}_1}{\Gamma \vdash M_1 = M_1' : \Pi x{\star}A.B} \quad \overset{\mathcal{D}_2}{\Gamma \vdash M_2 = M_2' \star A}}{\Gamma \vdash M_1 * M_2 = M_1' * M_2' : [M_2/x]B}\ \mathsf{oe\_app}$$

The induction hypotheses allow us to conclude

$$\Gamma \vdash M_1, M_1' : \Pi x{\star}A.B$$

$$\Gamma \vdash \Pi x{\star}A.B : \mathsf{type}$$

$$\Gamma \vdash M_2, M_2' \star A$$

$$\Gamma \vdash A : \mathsf{type}$$

So $\Gamma \vdash M_1 * M_2 : [M_2/x]B$ follows immediately from $\mathsf{ot\_app}$. By inversion of products, $\Gamma, x \star A \vdash B : \mathsf{type}$. By substitution, $\Gamma \vdash [M_2/x]B : \mathsf{type}$, and by functionality, $\Gamma \vdash [M_2/x]B = [M_2'/x]B : \mathsf{type}$. Therefore we can form a derivation

$$\frac{\dfrac{\begin{array}{c}\Gamma \vdash M_1' : \Pi x{\star}A.B \\ \Gamma \vdash M_2' \star A\end{array}}{\Gamma \vdash M_1' * M_2' : [M_2'/x]B}\ \mathsf{ot\_app} \quad \dfrac{\Gamma \vdash [M_2/x]B = [M_2'/x]B : \mathsf{type}}{\Gamma \vdash [M_2'/x]B = [M_2/x]B : \mathsf{type}}\ \mathsf{tv\_sym}}{\Gamma \vdash M_1' * M_2' : [M_2/x]B}\ \mathsf{ot\_conv}$$

Case:

$$\frac{\overset{\mathcal{D}_1}{\{\Gamma \vdash A : \mathsf{type}\}} \quad \overset{\mathcal{D}_2}{\Gamma, x \star A \vdash M_2 = M_2' : B} \quad \overset{\mathcal{D}_3}{\Gamma \vdash M_1 = M_1' \star A}}{\Gamma \vdash (\lambda x{\star}A.M_2) * M_1 = [M_1'/x]M_2' : [M_1/x]B}\ \mathsf{oe\_red}$$

By the induction hypothesis on $\mathcal{D}_2$ and $\mathcal{D}_3$ we have $\Gamma, x \star A \vdash M_2, M_2' : B$, $\Gamma, x \star A \vdash B : \mathsf{type}$, and $\Gamma \vdash M_1, M_1' \star A$. Immediately we can show

$$\frac{\dfrac{\overset{D_1}{\Gamma \vdash A : \mathsf{type}} \quad \Gamma, x \star A \vdash M_2 : B}{\Gamma \vdash \lambda x{\star}A.M_2 : \Pi x{\star}A.B}\ \mathsf{ot\_lam} \quad \Gamma \vdash M_1 \star A}{\Gamma \vdash (\lambda x{\star}A.M_2) * M_1 : [M_1/x]B}\ \mathsf{ot\_app}$$

and apply substitution to see that $\Gamma \vdash [M_1/x]B : \mathsf{type}$. Substitution also provides $\Gamma \vdash [M_1'/x]M_2' : [M_1'/x]B$, so we again use functionality, symmetry of type equality, and type conversion to obtain $\Gamma \vdash [M_1'/x]M_2' : [M_1/x]B$.

Case:

$$\mathcal{D} = \cfrac{\overset{\mathcal{D}_1}{\Gamma^{\oplus} \vdash M = M : A} \quad \overset{\mathcal{D}_2}{\Gamma^{\oplus} \vdash N = N : A} \quad \overset{\mathcal{D}_3}{\Gamma \vdash A : \mathsf{type}}}{\Gamma \vdash M = N \div A}\,\mathsf{pe}$$

The induction hypothesis gives $\mathcal{D}_1' :: \Gamma^{\oplus} \vdash M : A$, $\mathcal{D}_2' :: \Gamma^{\oplus} \vdash N : A$. Therefore

$$\cfrac{\overset{\mathcal{D}_1'}{\Gamma^{\oplus} \vdash M : A} \quad \overset{\mathcal{D}_3}{\Gamma \vdash A : \mathsf{type}}}{\Gamma \vdash M \div A}\,\mathsf{pot}$$

and

$$\cfrac{\overset{\mathcal{D}_2'}{\Gamma^{\oplus} \vdash N : A} \quad \overset{\mathcal{D}_3}{\Gamma \vdash A : \mathsf{type}}}{\Gamma \vdash N \div A}\,\mathsf{pot}$$

and immediately $\Gamma \vdash A : \mathsf{type}$.

■

Lemma 7.3 (Functionality for Equality) *Suppose* $\vdash \Gamma, x \star A : \mathsf{ctx}$ *and* $\Gamma \vdash M = N \star A$. *Then*

1. *If* $\Gamma, x \star A \vdash O = P : B$, *then* $\Gamma \vdash [M/x]O = [N/x]P : [M/x]B$.
2. *If* $\Gamma, x \star A \vdash B = C : K$, *then* $\Gamma \vdash [M/x]B = [N/x]C : [M/x]K$.
3. *If* $\Gamma, x \star A \vdash K = L : \mathsf{kind}$, *then* $\Gamma \vdash [M/x]K = [N/x]L : \mathsf{kind}$.

Proof Direct. We show the first part; the other two are similar.

$\Gamma, x \star A \vdash O = P : B$	Assumption.
$\Gamma \vdash M = N \star A$	Assumption.
$\Gamma \vdash M \star A$	Validity.
$\Gamma \vdash N \star A$	Validity.
$\Gamma \vdash [M/x]O = [M/x]P : [M/x]B$	Substitution.
$\Gamma, x \star A \vdash P : B$	Validity.
$\Gamma \vdash [M/x]P = [N/x]P : [M/x]B$	Functionality.
$\Gamma \vdash [M/x]O = [N/x]P : [M/x]B$	oe_trans.

■

Lemma 7.4 (Typing Inversion) *Suppose* $\vdash \Gamma : \mathsf{ctx}$. *Then*

1. *If* $\Gamma \vdash x : A$ *then* $x : B \in \Gamma$ *and* $\Gamma \vdash A = B : \mathsf{type}$ *for some* B.
2. *If* $\Gamma \vdash c : A$ *then* $c : B \in \Sigma$ *and* $\Gamma \vdash A = B : \mathsf{type}$ *for some* B.
3. *If* $\Gamma \vdash M_1 * M_2 : A$ *then* $\Gamma \vdash M_1 : \Pi x{\star}A_2.A_1$, $\Gamma \vdash M_2 \star A_2$, *and* $\Gamma \vdash [M_2/x]A_1 = A : \mathsf{type}$ *for some* A_1 *and* A_2.

4. *If* $\Gamma \vdash \lambda x{\star}A.M : B$, *then* $\Gamma \vdash B = \Pi x{\star}A.A' : \mathsf{type}$, $\Gamma \vdash A : \mathsf{type}$, *and* $\Gamma, x \star A \vdash M : A'$.

5. *If* $\Gamma \vdash \Pi x{\star}A_1.A_2 : K$, *then* $\Gamma \vdash K = \mathsf{type} : \mathsf{kind}$, $\Gamma \vdash A_1 : \mathsf{type}$, *and* $\Gamma, x \star A_1 \vdash A_2 : \mathsf{type}$.

6. *If* $\Gamma \vdash a : K$, *then* $a : L \in \Sigma$ *and* $\Gamma \vdash K = L : \mathsf{kind}$ *for some* L.

7. *If* $\Gamma \vdash A{*}M : K$ *then* $\Gamma \vdash A : \Pi x{\star}A_1.K_2$, $\Gamma \vdash M{\star}A_1$, *and* $\Gamma \vdash [M/x]K_2 = K : \mathsf{kind}$ *for some* K_1 *and* K_2.

8. *If* $\Gamma \vdash \Pi x{\star}A_1.K : \mathsf{kind}$, *then* $\Gamma \vdash A_1 : \mathsf{type}$, *and* $\Gamma, x \star A_1 \vdash K : \mathsf{kind}$.

Proof Straightforward induction, using validity to be able to apply reflexivity. ■

Lemma 7.5 (Redundancy) *The typing premises in rules* oe_red, te_pi, *and* ke_pi *enclosed in* $\{$*braces*$\}$ *are redundant.*

Proof Straightforward from validity. ■

Lemma 7.6 (Equality Inversion)

1. *If* $\Gamma \vdash A = \Pi x{\star}B_1.B_2 : \mathsf{type}$ *or* $\Gamma \vdash \Pi x{\star}B_1.B_2 = A : \mathsf{type}$, *then* A *is of the form* $\Pi x{\star}A_1.A_2$ *such that* $\Gamma \vdash A_1 = B_1 : \mathsf{type}$ *and* $\Gamma, x : A_1 \vdash A_2 = B_2 : \mathsf{type}$.

2. *If* $\Gamma \vdash K = \mathsf{type} : \mathsf{kind}$ *or* $\Gamma \vdash \mathsf{type} = K : \mathsf{kind}$, *then* K *is* type.

3. *If* $\Gamma \vdash K = \Pi x{\star}B_1.L_2 : \mathsf{kind}$ *or* $\Gamma \vdash \Pi x{\star}B_1.L_2 = K : \mathsf{kind}$, *then* K *is of the form* $\Pi x{\star}A_1.K_2$ *such that* $\Gamma \vdash A_1 = B_1 : \mathsf{type}$ *and* $\Gamma, x : A_1 \vdash K_2 = L_2 : \mathsf{type}$.

Proof By induction. ■

Lemma 7.7 (Injectivity of Products)

1. *If* $\Gamma \vdash \Pi x{\star}A_1.A_2 = \Pi x{\star}B_1.B_2 : \mathsf{type}$ *then* $\Gamma \vdash A_1 = B_1 : \mathsf{type}$ *and* $\Gamma, x \star A_1 \vdash A_2 = B_2 : \mathsf{type}$.

2. *If* $\Gamma \vdash \Pi x{\star}A.K = \Pi x{\star}B.L : \mathsf{kind}$ *then* $\Gamma \vdash A = B : \mathsf{type}$ *and* $\Gamma, x \star A \vdash K = L : \mathsf{kind}$.

Proof Immediate from Lemma 7.6. ■

8 Algorithmic Equality

Simple Kinds	κ	$::= \mathsf{type}^- \mid \tau \to \kappa \mid \tau \stackrel{\div}{\to} \kappa$
Simple Types	τ	$::= \alpha \mid \tau_1 \to \tau_2 \mid \tau_1 \stackrel{\div}{\to} \tau_2$
Simple Contexts	Δ	$::= \cdot \mid \Delta, x \star \tau$

The symbol $\stackrel{*}{\to}$ denotes either $\to$ or $\stackrel{\div}{\to}$.

Functions:

Erasure	$A \mapsto A^-$	Map types to simple types

Judgments:

Weak Head Reduction	$M_1 \stackrel{whr}{\longrightarrow} M_2$	Σ is a valid signature.
Algorithmic Equality	$M_1 \Longleftrightarrow M_2 : \tau$	M_1 is algorithmically equal to M_2 at simple type τ
Structural Equality	$M_1 \longleftrightarrow M_2 : \tau$	M_1 is structurally equal to M_2 at simple type τ

8.1 Erasure

$$\begin{array}{rcl} (a)^- & = & a \\ (A * M)^- & = & A^- \\ (\Pi x{\star}A_1.A_2) & = & A_1^- \stackrel{*}{\to} A_2^- \\ (\Pi x{\star}A.K) & = & A^- \stackrel{*}{\to} K^- \\ (\Gamma, x : A)^- & = & \Gamma^-, x : A^- \end{array}$$

8.2 Algorithmic Equality

8.2.1 Weak Head Reduction

$$\frac{}{(\lambda x{\star}A_1.M_2)M_1 \stackrel{whr}{\longrightarrow} [M_1/x]M_2}\ \mathsf{whr\_beta}$$

$$\frac{M_1 \stackrel{whr}{\longrightarrow} M_1'}{M_1 * M_2 \stackrel{whr}{\longrightarrow} M_1' * M_2}\ \mathsf{whr\_head}$$

8.2.2 Algorithmic Object Equality

$$\frac{M \stackrel{whr}{\longrightarrow} M' \Delta \vdash M' \Longleftrightarrow N : \alpha}{\Delta \vdash M \Longleftrightarrow N : \alpha}\ \mathsf{ae\_whrl}$$

$$\frac{N \stackrel{whr}{\longrightarrow} N' \Delta \vdash M \Longleftrightarrow N' : \alpha}{\Delta \vdash M \Longleftrightarrow N : \alpha}\ \mathsf{ae\_whrr}$$

$$\frac{\Delta, x \star \tau_1 \vdash M * x \Longleftrightarrow N * x : \tau_2}{\Delta \vdash M \Longleftrightarrow N : \tau_1 \stackrel{*}{\rightarrow} \tau_2} \text{ae\_ext}$$

$$\frac{\Delta \vdash M \longleftrightarrow N : \alpha}{\Delta \vdash M \Longleftrightarrow N : \alpha} \text{ae\_str}$$

8.2.3 Structural Object Equality

$$\frac{x : \tau \in \Delta}{\Delta \vdash x = x : \tau} \text{se\_var}$$

$$\frac{c : A \in \Sigma}{\Delta \vdash c = c : A^-} \text{se\_const}$$

$$\frac{\Delta \vdash M_1 \longleftrightarrow N_1 : \tau_2 \rightarrow \tau_1 \qquad \Delta \vdash M_2 \Longleftrightarrow N_2 : \tau_2}{\Delta \vdash M_1\ M_2 \longleftrightarrow N_1\ N_2 : \tau_1} \text{se\_app}$$

$$\frac{\Delta \vdash M_1 \longleftrightarrow N_1 : \tau_2 \rightarrow \tau_1}{\Delta \vdash M_1 * M_2 \longleftrightarrow N_1 * N_2 : \tau_1} \text{se\_iapp}$$

8.2.4 Algorithmic Family Equality

$$\frac{\Delta, x \star \tau \vdash A * x \Longleftrightarrow B * x : \kappa}{\Delta \vdash A \Longleftrightarrow B : \tau \stackrel{*}{\rightarrow} \kappa} \text{afe\_ext}$$

$$\frac{\Delta \vdash A \longleftrightarrow B : \mathsf{type}^-}{\Delta \vdash A \Longleftrightarrow B : \mathsf{type}^-} \text{afe\_str}$$

$$\frac{\Delta \vdash A_1 \Longleftrightarrow B_1 : \mathsf{type}^- \qquad \Delta, x \star A_1 \vdash A_2 \Longleftrightarrow B_2 : \mathsf{type}^-}{\Delta \vdash \Pi x{\star}A_1.A_2 \Longleftrightarrow \Pi x{\star}B_1.B_2 : \mathsf{type}^-} \text{afe\_pi}$$

8.2.5 Structural Family Equality

$$\frac{a : K \in \Sigma}{\Delta \vdash a = a : K^-} \text{sfe\_const}$$

$$\frac{\Delta \vdash A \longleftrightarrow B : \tau \rightarrow \kappa \qquad \Delta \vdash M \Longleftrightarrow N : \tau}{\Delta \vdash A\ M \longleftrightarrow B\ N : \kappa} \text{sfe\_app}$$

$$\frac{\Delta \vdash A \longleftrightarrow B : \tau \rightarrow \kappa}{\Delta \vdash A * M \longleftrightarrow B * N : \kappa} \text{sfe\_iapp}$$

8.2.6 Algorithmic Kind Equality

$$\frac{}{\Delta \vdash \mathsf{type} \Longleftrightarrow \mathsf{type} : \mathsf{kind}^-} \text{ake\_type}$$

$$\frac{\Delta \vdash A \Longleftrightarrow B : \mathsf{type}^- \qquad \Delta, x \star A \vdash K \Longleftrightarrow L : \mathsf{type}^-}{\Delta \vdash \Pi x{\star}A.K \Longleftrightarrow \Pi x{\star}B.L : \mathsf{type}^-} \text{ake\_pi}$$