Phase Semantics of Focused ILL

Jason Reed

November 11, 2015

1 Introduction

We show how the phase semantics for intuitionistic linear logic can be refined into a phase semantics for *focused* intuitionistic linear logic. The closure operator in the former arises from the Galois connection between upshift and downshift in the latter.

2 Language

We recall the syntax of focused intuitionistic linear logic. Propositions are polarized into positive and negative. There are shift operators $\uparrow$ and $\downarrow$ that coerce back and forth between the two polarities. Atomic propositions also come in positive a^+ and a^-. Somewhat unusually (but it's just a matter of presentation, not an essential part of the result) we distinguish an atomic proposition $a^\pm$ of either polarity from the suspension $\langle a^\pm \rangle$ of it that arises after asynchronous decomposition terminates at it.

$$\begin{array}{rcll}
\text{Positives} & P & ::= & {\downarrow}N \mid P \otimes P \mid P \oplus P \mid 1 \mid 0 \mid a^+ \\
\text{Negatives} & N & ::= & {\uparrow}P \mid P \multimap N \mid N \,\&\, N \mid \top \mid a^- \\
\text{Positive Contexts} & \Omega & ::= & \cdot \mid P, \Omega \\
\text{Negative Contexts} & \Gamma & ::= & \cdot \mid \Gamma, H \\
\text{Stable Hypotheses} & H & ::= & N \mid \langle a^+ \rangle \\
\text{Stable Conclusions} & Q & ::= & P \mid \langle a^- \rangle \\
\text{Conclusions} & R & ::= & N \mid Q
\end{array}$$

The three judgments of the logic are

$$\begin{array}{rl}
\text{Inversion} & \Gamma; \Omega \vdash R \\
\text{Right Focus} & \Gamma \vdash [P] \\
\text{Left Focus} & \Gamma[N] \vdash Q
\end{array}$$

(we sometimes abbreviate $\Gamma; \cdot \vdash R$ as $\Gamma \vdash R$) and the proof rules for the focusing system are in Figure 1.

$$\frac{\Gamma; P \vdash N}{\Gamma; \cdot \vdash P \multimap N} \multimap R \qquad \frac{\Gamma_1 \vdash [P] \qquad \Gamma_2[N] \vdash Q}{\Gamma_1, \Gamma_2[P \multimap N] \vdash Q} \multimap L \qquad \frac{\Gamma; \cdot \vdash N_1 \qquad \Gamma; \cdot \vdash N_2}{\Gamma; \cdot \vdash N_1 \,\&\, N_2} \&R$$

$$\frac{\Gamma[N_i] \vdash Q}{\Gamma[N_1 \,\&\, N_2] \vdash Q} \&L \qquad \frac{\Gamma_1 \vdash [P_1] \qquad \Gamma_2 \vdash [P_2]}{\Gamma_1, \Gamma_2 \vdash [P_1 \otimes P_2]} \otimes R \qquad \frac{\Gamma; P_1, P_2, \Omega \vdash R}{\Gamma; P_1 \otimes P_2, \Omega \vdash R} \otimes L$$

$$\frac{\Gamma \vdash [P_i]}{\Gamma \vdash [P_1 \oplus P_2]} \oplus R_i \qquad \frac{\Gamma; P_1, \Omega \vdash R \qquad \Gamma; P_2, \Omega \vdash R}{\Gamma; P_1 \oplus P_2, \Omega \vdash R} \oplus L \qquad \frac{}{\Gamma; \cdot \vdash \top} \top R \qquad \frac{}{\Delta \vdash [1]} 1R$$

$$\frac{\Gamma; \Omega \vdash R}{\Gamma; 1, \Omega \vdash R} 1L \qquad \frac{}{\Gamma; 0, \Omega \vdash R} 0L \qquad \frac{\Gamma; \cdot \vdash N}{\Gamma \vdash [{\downarrow}N]} {\downarrow}R \qquad \frac{\Gamma, N; \Omega \vdash R}{\Gamma; {\downarrow}N, \Omega \vdash R} {\downarrow}L \qquad \frac{\Gamma; \cdot \vdash P}{\Gamma; \cdot \vdash {\uparrow}P} {\uparrow}R$$

$$\frac{\Gamma; P \vdash Q}{\Gamma[{\uparrow}P] \vdash Q} {\uparrow}L \qquad \frac{}{\langle a^+ \rangle \vdash [a^+]} a^+R \qquad \frac{\Gamma, \langle a^+ \rangle; \Omega \vdash R}{\Gamma; a^+, \Omega \vdash R} a^+L \qquad \frac{\Gamma; \cdot \vdash \langle a^- \rangle}{\Gamma; \cdot \vdash a^-} a^-R$$

$$\frac{}{[a^-] \vdash \langle a^- \rangle} a^-L \qquad \frac{\Gamma \vdash [P]}{\Gamma; \cdot \vdash P} focR \qquad \frac{\Gamma[N] \vdash Q}{\Gamma, N; \cdot \vdash Q} focL$$

Figure 1: Focused Linear Logic Proof Rules

3 Semantics

We now interpret this logic in *phase monoid actions*. A phase monoid action is a tuple $(M, X, S, \otimes, 1, \multimap)$ where

- $(M, \otimes, 1)$ is a commutative monoid
- X is a set, and $\multimap\colon M \times X \to X$ is a monoid action of M on X
- S is a distinguished subset of X

Given a mapping η of positive atoms a^+ to subsets of M and from negative atoms a^- to subsets of X, we inductively define an interpretation of all positive propositions as subsets of M, and negative propositions as subsets of X.

$$[\![a^\pm]\!] = \eta(a^\pm) \qquad [\![P \multimap N]\!] = [\![P]\!] \multimap [\![N]\!]$$
$$[\![P_1 \otimes P_2]\!] = [\![P_1]\!] \otimes [\![P_2]\!] \qquad [\![1]\!] = \{1\}$$
$$[\![P_1 \oplus P_2]\!] = [\![P_1]\!] \cup [\![P_2]\!] \qquad [\![0]\!] = \emptyset$$
$$[\![N_1 \,\&\, N_2]\!] = [\![N_1]\!] \cup [\![N_2]\!] \qquad [\![\top]\!] = \emptyset$$
$$[\![{\uparrow}P]\!] = \{n \in X \mid \forall p \in [\![P]\!].\, (p \multimap n) \in S\}$$
$$[\![{\downarrow}N]\!] = \{p \in M \mid \forall n \in [\![N]\!].\, (p \multimap n) \in S\}$$

where $\otimes$ and $\multimap$ are overloaded as set operators in the evident direct-image sort of way, i.e.

$$M_1 \otimes M_2 = \{p_1 \otimes p_2 \mid p_1 \in M_1, p_2 \in M_2\}$$

and

$$M' \multimap X' = \{p \multimap n \mid p \in M', n \in X'\}$$

where $M', M_1, M_2 \subseteq M, X' \subseteq X$.

We say $\models P$ if for every phase monoid action, and every interpretation η of atoms in that action, we have $1 \in [\![P]\!]$.

4 Soundness

Make the following notational conventions:

- $[\![Q]\!]$ is $[\![\uparrow P]\!]$ if $Q = P$, and $\eta(a^-)$ if $Q = \langle a^- \rangle$.
- $[\![H]\!]$ is $[\![\downarrow N]\!]$ if $H = N$, and $\eta(a^+)$ if $H = \langle a^+ \rangle$.
- $[\![P_1, \ldots, P_n]\!] = [\![P_1]\!] \otimes \cdots \otimes [\![P_n]\!]$
- $[\![H_1, \ldots, H_n]\!] = [\![H_1]\!] \otimes \cdots \otimes [\![H_n]\!]$

Theorem 4.1

1. *If* $\Gamma; \Omega \vdash R$ *then* $[\![\Gamma]\!] \otimes [\![\Omega]\!] \multimap [\![R]\!] \subseteq S$
2. *If* $\Gamma \vdash [P]$ *then* $[\![\Gamma]\!] \subseteq [\![P]\!]$
3. *If* $\Gamma[N] \vdash Q$ *then* $[\![\Gamma]\!] \multimap [\![Q]\!] \subseteq [\![N]\!]$

Proof By induction on the proof, unpacking definitions and checking equality in a fairly straightforward way. ■

Corollary 4.2 (Soundness) *If* $\vdash [P]$*, then* $\models P$*.*

Proof By part 2, $\vdash [P]$ immediately gives $1 \in [\![P]\!]$. ■

5 Completeness

We first recall that the focusing proof system is well-formed; it satisfies cut elimination and an identity principle.

Lemma 5.1 (Cut)

1. *If* $\Gamma_1[N] \vdash Q$ *and* $\Gamma_2 \vdash N$*, then* $\Gamma_1, \Gamma_2 \vdash Q$*.*
2. *If* $\Gamma_1 \vdash [P]$ *and* $\Gamma_2; P \vdash Q$*, then* $\Gamma_1, \Gamma_2 \vdash Q$*.*

Proof Standard structural proof, which we omit; generalize the induction hypothesis slightly to cover commutative cases, and proceed by induction on the cut proposition and derivations. ∎

Lemma 5.2 (Identity)

1. $N \vdash N$
2. $P \vdash P$
3. *If* $\Gamma_0[N] \vdash Q$ *implies* $\Gamma_0, \Gamma_\bullet \vdash Q$ *for every* Γ_0 *and* Q*, then* $\Gamma_\bullet \vdash N$.
4. *If* $\Gamma_0 \vdash [P]$ *implies* $\Gamma_0, \Gamma_\bullet; \Omega \vdash R$ *for every* Γ_0*, then* $\Gamma_\bullet; P, \Omega \vdash R$.

Proof Only slightly nonstandard. The important idea is thinking in terms of suspensions as in Simmons's "Structural focalization".

By induction on P and N, and the case of the theorem. To see $N \vdash N$, appeal to part 3 with $\Gamma_\bullet = N$, and use rule $focL$. To see $P \vdash P$, appeal to part 4 with $\Gamma_\bullet = \cdot$ and $Q = P$, and use rule $focR$.

To show parts 3 and 4, we split cases on N and P:

Case: $N = P \multimap N_0$. We know that $\Gamma_0[P \multimap N_0] \vdash Q$ implies $\Gamma_0, \Gamma_\bullet \vdash Q$ for every Γ_0 and Q. Call this fact $(*)$. We must show $\Gamma_\bullet \vdash P \multimap N_0$, and so we try to construct $\Gamma_\bullet; P \vdash N_0$. By induction hypothesis, it suffices to show for every Γ_+ that $\Gamma_+ \vdash [P]$ implies $\Gamma_+, \Gamma_\bullet \vdash N$. By induction hypothesis on N, it suffices to show that $\Gamma_-[N] \vdash Q$ implies $\Gamma_-, \Gamma_+, \Gamma_\bullet \vdash Q$ for every Γ_-, Q. But now that we have $\Gamma_-[N] \vdash Q$ and $\Gamma_+ \vdash [P]$ we can construct $\Gamma_-, \Gamma_+[P \multimap N] \vdash Q$ and conclude $\Gamma_-, \Gamma_+, \Gamma_\bullet \vdash Q$ by $(*)$ with $\Gamma_0 = \Gamma_-, \Gamma_+$.

Case: $P = P_1 \otimes P_2$. We assume that $\Gamma_0 \vdash [P_1 \otimes P_2]$ implies $\Gamma_0, \Gamma_\bullet; \Omega \vdash R$ for every Γ_0. Call this fact $(*)$. We must show $\Gamma_\bullet; P_1 \otimes P_2, \Omega \vdash R$, and so we try to construct $\Gamma_\bullet; P_1, P_2, \Omega \vdash R$. By induction hypothesis on P_1, it suffices to show for every Γ_1 that $\Gamma_1 \vdash [P_1]$ implies $\Gamma_1, \Gamma_\bullet; P_2, \Omega \vdash R$. By induction hypothesis on P_2, it suffices to show for every Γ_2 that $\Gamma_2 \vdash [P_2]$ implies $\Gamma_1, \Gamma_2, \Gamma_\bullet; \Omega \vdash R$. But at this point, knowing what we know, we can prove $\Gamma_1, \Gamma_2 \vdash [P_1 \otimes P_2]$ and invoke $(*)$ with $\Gamma_0 = \Gamma_1, \Gamma_2$ to obtain $\Gamma_1, \Gamma_2, \Gamma_\bullet; \Omega \vdash R$ as required.

Case: $P = 1$. We assume that $\Gamma_0 \vdash [1]$ implies $\Gamma_0, \Gamma_\bullet; \Omega \vdash R$ for every Γ_0. In particular $\vdash [1]$ so we have $\Gamma_\bullet; \Omega \vdash R$, from which we can derive $\Gamma_\bullet; 1, \Omega \vdash R$, as required.

Case: $P = P_1 \oplus P_2$. We assume that $\Gamma_0 \vdash [P_1 \oplus P_2]$ implies $\Gamma_0, \Gamma_\bullet; \Omega \vdash R$ for every Γ_0. Call this fact $(*)$. We must show $\Gamma_\bullet; P_1 \oplus P_2, \Omega \vdash R$, and so we try to construct $\Gamma_\bullet; P_i, \Omega \vdash R$ for both $i \in \{1, 2\}$. By induction hypothesis on P_i, it suffices to show for every Γ that $\Gamma \vdash [P_i]$ implies $\Gamma, \Gamma_\bullet; \Omega \vdash R$. But having assumed $\Gamma \vdash [P_i]$, we can also show $\Gamma \vdash [P_1 \oplus P_2]$. This lets us apply fact $(*)$ to see $\Gamma, \Gamma_\bullet; \Omega \vdash R$ as required.

Case: $P = 0$. We can trivially prove $\Gamma_\bullet; 0, \Omega \vdash R$.

Case: $N = N_1 \mathbin{\&} N_2$. We assume that $\Gamma_0[N_1 \mathbin{\&} N_2] \vdash Q$ implies $\Gamma_0, \Gamma_\bullet \vdash Q$ for every Γ_0 and Q. Call this fact $(*)$. We must show $\Gamma_\bullet \vdash N_1 \mathbin{\&} N_2$, and so we try to construct $\Gamma_\bullet \vdash N_i$ for both $i \in \{1, 2\}$. By induction hypothesis on N_i, it suffices to show for every Γ, Q' that $\Gamma[N_i] \vdash Q'$ implies $\Gamma, \Gamma_\bullet \vdash Q'$. But having assumed $\Gamma[N_i] \vdash Q'$, we can also show $\Gamma[N_1 \mathbin{\&} N_2] \vdash Q'$. This lets us apply fact $(*)$ to see $\Gamma, \Gamma_\bullet \vdash Q'$ as required.

Case: $N = \top$. We can trivially prove $\Gamma_\bullet \vdash \top$.

Case: $N = a^-$. We assume that $\Gamma_0[a^-] \vdash Q$ implies $\Gamma_0, \Gamma_\bullet \vdash Q$ for every Γ_0 and Q. In particular $[a^-] \vdash \langle a^- \rangle$ so we have $\Gamma_\bullet \vdash \langle a^- \rangle$, from which we can derive $\Gamma_\bullet \vdash a^-$, as required.

Case: $P = a^+$. We assume that $\Gamma_0 \vdash [a^+]$ implies $\Gamma_0, \Gamma_\bullet; \Omega \vdash R$ for every Γ_0. In particular $\langle a^+ \rangle \vdash [a^+]$ so we have $\Gamma_\bullet, \langle a^+ \rangle; \Omega \vdash R$, from which we can derive $\Gamma_\bullet; a^+, \Omega \vdash R$, as required.

Case: $N = \uparrow P$. We assume that $\Gamma_0[\uparrow P] \vdash Q$ implies $\Gamma_0, \Gamma_\bullet \vdash Q$ for every Γ_0 and Q. By induction hypothesis $P \vdash P$, so also $[\uparrow P] \vdash P$, hence $\Gamma_\bullet \vdash P$, from which we can derive $\Gamma_\bullet \vdash \uparrow P$, as required.

Case: $P = \downarrow N$. We assume that $\Gamma_0 \vdash [\downarrow N]$ implies $\Gamma_0, \Gamma_\bullet; \Omega \vdash R$ for every Γ_0. By induction hypothesis $N \vdash N$ so also $N \vdash [\downarrow N]$, hence $\Gamma_\bullet, N; \Omega \vdash R$, from which we can derive $\Gamma_\bullet; \downarrow N, \Omega \vdash R$, as required. ■

Theorem 5.3 (Completeness) *If* $\models P$ *then* $\vdash P$.

Proof We build a universal model. Let M be the set of all negative contexts Γ, with $\otimes$ being multiset union and 1 being the empty context. Let X be the set of all pairs (Γ, Q). The context monoid M acts on X by concatenating onto the context: $\Gamma_1 \multimap (\Gamma_2, Q) = ((\Gamma_1, \Gamma_2), Q)$. The set S is $\{(\Gamma, Q) \in X \mid \Gamma \vdash Q\}$.

We choose as interpretations of the atoms $\eta(a^+) = \{\Gamma \mid \langle a^+ \rangle \in \Gamma\}$ and $\eta(a^-) = \{(\Gamma, \langle a^- \rangle) \mid \forall \Gamma\}$ The main lemma is showing that this model is in fact universal — that the interpretation of every proposition reflects its (focused) provability. We claim that for all P and N we have

- $[\![P]\!] = \{\Gamma \mid \Gamma \vdash [P]\}$
- $[\![N]\!] = \{(\Gamma, Q) \mid \Gamma[N] \vdash Q\}$

Clearly this suffices for the theorem, since if $\models P$ then $1 \in \{\Gamma \mid \Gamma \vdash [P]\}$ means $\vdash [P]$ and therefore $\vdash P$. We proceed by induction on the structure of P or N. Most of the cases are extremely easy:

Case: $P = a^+$. By inspection of the proof rules, $\{\Gamma \mid \Gamma \vdash [a^+]\}$ is those Γ that already include $\langle a^+ \rangle$ as an element, which is just what we chose $\eta(a^+)$ to be.

Case: $P = a^-$. Similarly, by inspection of the proof rules, $\{(\Gamma, Q) \mid \Gamma[a^-] \vdash Q\}$ demands that $Q = \langle a^- \rangle$ and puts no restriction on Γ, just as we did when defining $\eta(a^-)$.

Case: $P = 0$. There are no Γ that have $\Gamma \vdash [\![0]\!]$ and $[\![0]\!]$ is empty.

Case: $N = \top$. There are no Γ, Q that have $\Gamma[\top] \vdash Q$ and $[\![\top]\!]$ is empty.

Case: $P = P_1 \oplus P_2$. By i.h. we know

$$[\![P_i]\!] = \{\Gamma \mid \Gamma \vdash [P_i]\}$$

and furthermore

$$\{\Gamma \mid \Gamma \vdash [P_1 \oplus P_2]\} = \{\Gamma \mid \Gamma \vdash [P_1]\} \cup \{\Gamma \mid \Gamma \vdash [P_2]\}$$

by inspection of the proof rules, so we are done.

Case: $N = N_1 \mathbin{\&} N_2$. By i.h. we know

$$[\![N_i]\!] = \{\Gamma, Q \mid \Gamma[N_i] \vdash Q\}$$

and furthermore

$$\{\Gamma, Q \mid \Gamma[N_1 \mathbin{\&} N_2] \vdash Q\} = \{\Gamma, Q \mid \Gamma[N_1] \vdash Q\} \cup \{\Gamma, Q \mid \Gamma[N_2] \vdash Q\}$$

by inspection of the proof rules, so we are done.

Case: $N = P \multimap N_0$. By i.h. we know

$$[\![P]\!] = \{\Gamma_1 \mid \Gamma_1 \vdash [P]\}$$

$$[\![N_0]\!] = \{\Gamma_2, Q \mid \Gamma_2[N_0] \vdash Q\}$$

so we can reason that

$$\begin{aligned}
&\{\Gamma, Q \mid \Gamma[P \multimap N_0] \vdash Q\} \\
&= \{(\Gamma_1, \Gamma_2), Q \mid \Gamma_1 \vdash [P] \wedge \Gamma_2[N_0] \vdash Q\} \\
&= \{(\Gamma_1, \Gamma_2), Q \mid \Gamma_1 \in [\![P]\!] \wedge (\Gamma_2, Q) \in [\![N_0]\!]\} \\
&= [\![P \multimap N_0]\!]
\end{aligned}$$

Case: $P = P_1 \otimes P_2$. By i.h. we know

$$[\![P_i]\!] = \{\Gamma_i \mid \Gamma_i \vdash [P_i]\}$$

so we can reason that

$$\begin{aligned}
&\{\Gamma \mid \Gamma \vdash [P_1 \otimes P_2]\} \\
&= \{(\Gamma_1, \Gamma_2) \mid \Gamma_1 \vdash [P_1] \wedge \Gamma_2 \vdash [P_2]\} \\
&= \{(\Gamma_1, \Gamma_2) \mid \Gamma_1 \in [\![P_1]\!] \wedge \Gamma_2 \in [\![P_2]\!]\} \\
&= [\![P_1 \otimes P_2]\!]
\end{aligned}$$

Case: $P = 1$. The only context that proves 1 in focus is the empty context which we chose as the monoid unit, so $[\![1]\!] = \{1\}$ is as required.

The remaining cases are the shifts, which are dealt with by appealing to harmony:

Case: $P = {\downarrow}N$. Here we have to show that $[\![{\downarrow}N]\!] = \{\Gamma \mid \Gamma \vdash [{\downarrow}N]\}$, in other words that

$$\{\Gamma_1 \mid \forall(\Gamma_2, Q) \in [\![N]\!].\Gamma_1, \Gamma_2 \vdash Q\} = \{\Gamma \mid \Gamma \vdash N\}$$

By induction hypothesis, this is the same as showing that

$$\{\Gamma_1 \mid \forall(\Gamma_2, Q).\Gamma_2[N] \vdash Q \Rightarrow \Gamma_1, \Gamma_2 \vdash Q\} = \{\Gamma_1 \mid \Gamma_1 \vdash N\}$$

To see $\supseteq$, use cut, and to see $\subseteq$, use identity.

Case: $N = {\uparrow}P$. Here we have to show that $[\![{\uparrow}P]\!] = \{\Gamma, Q \mid \Gamma[{\uparrow}P] \vdash Q\}$, in other words that

$$\{\Gamma_1, Q \mid \forall\Gamma_2 \in [\![P]\!].\Gamma_1, \Gamma_2 \vdash Q\} = \{\Gamma, Q \mid \Gamma; P \vdash Q\}$$

By induction hypothesis, this is the same as showing that

$$\{\Gamma_1, Q \mid \forall\Gamma_2.\Gamma_2 \vdash [P] \Rightarrow \Gamma_1, \Gamma_2 \vdash Q\} = \{\Gamma_1, Q \mid \Gamma_1; P \vdash Q\}$$

To see $\supseteq$, use cut, and to see $\subseteq$, use identity.

■