Categorical Semantics of Focused ILL

Jason Reed

November 30, 2017

1 Introduction

I want to describe a certain categorical semantics of focused intuitionistic linear logic so that I can better understand how it relates to other semantics I've seen in the literature [HS07].

2 Language

We recall the syntax of focused intuitionistic linear logic, leaving out exponentials for now. Propositions are polarized into positive and negative. There are shift operators $\uparrow$ and $\downarrow$ that coerce back and forth between the two polarities. Atomic propositions also come in positive a^+ and a^-. Somewhat unusually (but it's just a matter of presentation, not an essential part of the result) we distinguish an atomic proposition $a^\pm$ of either polarity from the suspension $\langle a^\pm \rangle$ of it that arises after asynchronous decomposition terminates at it.

$$\begin{array}{rccl}
\text{Positives} & P & ::= & \downarrow N \mid P \otimes P \mid P \oplus P \mid 1 \mid 0 \mid a^+ \\
\text{Negatives} & N & ::= & \uparrow P \mid P \multimap N \mid N \mathbin{\&} N \mid \top \mid a^- \\
\text{Positive Contexts} & \Omega & ::= & \cdot \mid P, \Omega \\
\text{Negative Contexts} & \Gamma & ::= & \cdot \mid \Gamma, H \\
\text{Stable Hypotheses} & H & ::= & N \mid \langle a^+ \rangle \\
\text{Stable Conclusions} & Q & ::= & P \mid \langle a^- \rangle \\
\text{Conclusions} & R & ::= & N \mid Q
\end{array}$$

The three judgments of the logic are

$$\begin{array}{rl}
\text{Inversion} & \Gamma; \Omega \vdash R \\
\text{Right Focus} & \Gamma \vdash [P] \\
\text{Left Focus} & \Gamma[N] \vdash Q
\end{array}$$

(we sometimes abbreviate $\Gamma; \cdot \vdash R$ as $\Gamma \vdash R$) and the proof rules for the focusing system are in Figure 1.

$$
\frac{\Gamma; P \vdash N}{\Gamma; \cdot \vdash P \multimap N}\,{\multimap}R
\qquad
\frac{\Gamma_1 \vdash [P] \qquad \Gamma_2[N] \vdash Q}{\Gamma_1, \Gamma_2[P \multimap N] \vdash Q}\,{\multimap}L
\qquad
\frac{\Gamma; \cdot \vdash N_1 \qquad \Gamma; \cdot \vdash N_2}{\Gamma; \cdot \vdash N_1 \mathbin{\&} N_2}\,\&R
$$

$$
\frac{\Gamma[N_i] \vdash Q}{\Gamma[N_1 \mathbin{\&} N_2] \vdash Q}\,\&L
\qquad
\frac{\Gamma_1 \vdash [P_1] \qquad \Gamma_2 \vdash [P_2]}{\Gamma_1, \Gamma_2 \vdash [P_1 \otimes P_2]}\,{\otimes}R
\qquad
\frac{\Gamma; P_1, P_2, \Omega \vdash R}{\Gamma; P_1 \otimes P_2, \Omega \vdash R}\,{\otimes}L
$$

$$
\frac{\Gamma \vdash [P_i]}{\Gamma \vdash [P_1 \oplus P_2]}\,{\oplus}R_i
\qquad
\frac{\Gamma; P_1, \Omega \vdash R \qquad \Gamma; P_2, \Omega \vdash R}{\Gamma; P_1 \oplus P_2, \Omega \vdash R}\,{\oplus}L
\qquad
\frac{}{\Gamma; \cdot \vdash \top}\,\top R
\qquad
\frac{}{\cdot \vdash [1]}\,1R
$$

$$
\frac{\Gamma; \Omega \vdash R}{\Gamma; 1, \Omega \vdash R}\,1L
\qquad
\frac{}{\Gamma; 0, \Omega \vdash R}\,0L
\qquad
\frac{\Gamma; \cdot \vdash N}{\Gamma \vdash [{\downarrow}N]}\,{\downarrow}R
\qquad
\frac{\Gamma, N; \Omega \vdash R}{\Gamma; {\downarrow}N, \Omega \vdash R}\,{\downarrow}L
\qquad
\frac{\Gamma; \cdot \vdash P}{\Gamma; \cdot \vdash {\uparrow}P}\,{\uparrow}R
$$

$$
\frac{\Gamma; P \vdash Q}{\Gamma[{\uparrow}P] \vdash Q}\,{\uparrow}L
\qquad
\frac{}{\langle a^+ \rangle \vdash [a^+]}\,a^+R
\qquad
\frac{\Gamma, \langle a^+ \rangle; \Omega \vdash R}{\Gamma; a^+, \Omega \vdash R}\,a^+L
\qquad
\frac{\Gamma; \cdot \vdash \langle a^- \rangle}{\Gamma; \cdot \vdash a^-}\,a^-R
$$

$$
\frac{}{[a^-] \vdash \langle a^- \rangle}\,a^-L
\qquad
\frac{\Gamma \vdash [P]}{\Gamma; \cdot \vdash P}\,focR
\qquad
\frac{\Gamma[N] \vdash Q}{\Gamma, N; \cdot \vdash Q}\,focL
$$

Figure 1: Focused Linear Logic Proof Rules

2.1 Results

These are thoroughly standard.

Lemma 2.1 (Cut)

1. *If* $\Gamma_1[N] \vdash Q$ *and* $\Gamma_2 \vdash N$*, then* $\Gamma_1, \Gamma_2 \vdash Q$*.*
2. *If* $\Gamma_1 \vdash [P]$ *and* $\Gamma_2; P \vdash Q$*, then* $\Gamma_1, \Gamma_2 \vdash Q$*.*

Lemma 2.2 (Identity)

1. $N \vdash N$
2. $P \vdash P$
3. *If* $\Gamma_0[N] \vdash Q$ *implies* $\Gamma_0, \Gamma \vdash Q$ *for every* Γ_0 *and* Q*, then* $\Gamma \vdash N$*.*
4. *If* $\Gamma_0 \vdash [P]$ *implies* $\Gamma_0, \Gamma; \Omega \vdash R$ *for every* Γ_0*, then* $\Gamma; P, \Omega \vdash R$*.*

3 Semantics

In what follows we write $\hat{\mathbf{C}}$ as an abbreviation for $\mathbf{C} \to \mathbf{Set}$, despite the usual contravariant convention.

A *model* M consists of the following data:

$$\frac{X \in \hat{\mathbf{P}} \qquad Y \in \hat{\mathbf{N}}}{X \mathbin{\hat{\multimap}} Y := \int^{\alpha:\mathbf{P},\phi:\mathbf{N}} X(\alpha) \times Y(\phi) \times \mathbf{N}(\alpha \multimap \phi, —)}$$

$$\frac{X \in \hat{\mathbf{P}} \qquad Y \in \hat{\mathbf{P}}}{X \mathbin{\hat{\otimes}} Y := \int^{\alpha_1:\mathbf{P},\alpha_2:\mathbf{P}} X(\alpha_1) \times Y(\alpha_2) \times \mathbf{P}(\alpha_1 \otimes \alpha_2, —)}$$

Figure 2: Semantic Operations

1. a symmetric monoidal category $(\mathbf{P}, \otimes, I)$ and a category $\mathbf{N}$
2. functors $\_ \multimap \_ : \mathbf{P} \times \mathbf{N} \to \mathbf{N}$ and $\_ \rhd \_ : \mathbf{P} \times \mathbf{N} \to \mathbf{Set}$
3. a natural isomorphism
$$k : \frac{(P_1 \otimes P_2) \rhd N}{P_1 \rhd (P_2 \multimap N)}$$
4. a mapping η of atoms a^+ (resp. a^-) to objects of $\hat{\mathbf{P}}$ (resp. $\hat{\mathbf{N}}$)

We inductively define an interpretation of all positive (resp. negative) propositions P (resp. N) as objects $[\![P]\!]_M \in \hat{\mathbf{P}}$ (resp. $[\![N]\!]_M \in \hat{\mathbf{N}}$). We write $[\![—]\!]$ instead of $[\![—]\!]_M$ when the M is evident from context. The definition of the interpretation is given in Figure 3, where $+$ denotes the (objectwise) coproduct and $\emptyset$ the initial object, in the category $\hat{P}$ or $\hat{N}$ as appropriate. The arrow $\to$ in the definition of the shifts is just the function space in **Set**. The multiplicative connectives are defined with coends, using the operators $\hat{\multimap}$ and $\hat{\otimes}$ defined in Figure 2. Shifts are defined with ends.

To interpret sequents, do as follows. If $X : \hat{\mathbf{C}}$ and $Y : \hat{\mathbf{D}}$, define their objectwise product $X \wedge Y : \widehat{\mathbf{C} \times \mathbf{D}}$ as $(X \wedge Y)(C, D) = X(C) \times Y(D)$.

We say

1. $\Gamma; \Omega \models_M R$ iff there is a nat. trans. $([\![\Gamma]\!] \mathbin{\hat{\otimes}} [\![\Omega]\!]) \wedge [\![R]\!]^{>} \stackrel{\cdot}{\to} \rhd$
2. $\Gamma \models_M [P]$ iff there is a nat. trans. $[\![\Gamma]\!] \stackrel{\cdot}{\to} [\![P]\!]$
3. $\Gamma[N] \models_M Q$ iff there is a nat. trans. $([\![\Gamma]\!] \mathbin{\hat{\multimap}} [\![Q]\!]^{>}) \stackrel{\cdot}{\to} [\![N]\!]$

When we drop the subscripts M and just write $\models$, it means that the statement holds for all M.

4 Results

Lemma 4.1 $\hat{\otimes}$ *is associative.*

$$[\![a^\pm]\!] = \eta(a^\pm) \qquad [\![1]\!] = \mathbf{P}(I, —)$$

$$[\![P_1 \oplus P_2]\!] = [\![P_1]\!] + [\![P_2]\!] \qquad [\![0]\!] = \emptyset$$

$$[\![N_1 \,\&\, N_2]\!] = [\![N_1]\!] + [\![N_2]\!] \qquad [\![\top]\!] = \emptyset$$

$$[\![P \multimap N]\!] = [\![P]\!] \mathbin{\hat{\multimap}} [\![N]\!]$$

$$[\![P_1 \otimes P_2]\!] = [\![P_1]\!] \mathbin{\hat{\otimes}} [\![P_2]\!]$$

$$[\![\uparrow P]\!] = \int_{\alpha:\mathbf{P}} [\![P]\!](\alpha) \to (\alpha \triangleright —)$$

$$[\![\downarrow N]\!] = \int_{\phi:\mathbf{N}} [\![N]\!](\phi) \to (— \triangleright \phi)$$

$$\frac{\Gamma = H_1, \ldots, H_n}{[\![\Gamma]\!] = [\![H_1]\!]^< \mathbin{\hat{\otimes}} \cdots \mathbin{\hat{\otimes}} [\![H_n]\!]^<} \qquad \frac{\Omega = P_1, \ldots, P_n}{[\![\Omega]\!] = [\![P_1]\!]^< \mathbin{\hat{\otimes}} \cdots \mathbin{\hat{\otimes}} [\![P_n]\!]^<}$$

$$[\![P]\!]^< = [\![P]\!] \qquad [\![N]\!]^< = [\![\downarrow N]\!] \qquad [\![\langle a^+ \rangle]\!]^< = \eta(a^+)$$

$$[\![N]\!]^> = [\![N]\!] \qquad [\![P]\!]^> = [\![\uparrow P]\!] \qquad [\![\langle a^- \rangle]\!]^> = \eta(a^-)$$

Figure 3: Interpreting Propositions

Proof

$$(X \mathbin{\hat{\otimes}} (Y \mathbin{\hat{\otimes}} Z))(\varepsilon) \cong \left(X \mathbin{\hat{\otimes}} \int^{\beta,\gamma:\mathbf{P}} Y(\beta) \times Z(\gamma) \times \mathbf{P}(\beta \otimes \gamma, —) \right)(\varepsilon)$$

$$\cong \int^{\alpha,\delta:\mathbf{P}} X(\alpha) \times \left(\int^{\beta,\gamma:\mathbf{P}} Y(\beta) \times Z(\gamma) \times \mathbf{P}(\beta \otimes \gamma, \delta) \right) \times \mathbf{P}(\alpha \otimes \delta, \varepsilon)$$

$$\cong \int^{\alpha,\beta,\delta,\gamma:\mathbf{P}} X(\alpha) \times Y(\beta) \times Z(\gamma) \times \mathbf{P}(\alpha \otimes \delta, \varepsilon) \times \mathbf{P}(\beta \otimes \gamma, \delta)$$

$$\cong \int^{\alpha,\beta,\delta:\mathbf{P}} X(\alpha) \times Y(\beta) \times Z(\gamma) \times \int^{\delta:\mathbf{P}} \mathbf{P}(\alpha \otimes \delta, \varepsilon) \times \mathbf{P}(\beta \otimes \gamma, \delta)$$

$$\cong \int^{\alpha,\beta,\gamma:\mathbf{P}} X(\alpha) \times Y(\beta) \times Z(\gamma) \times \mathbf{P}(\alpha \otimes \beta \otimes \gamma, \varepsilon)$$

and we can show symmetrically that

$$(X \mathbin{\hat{\otimes}} Y) \mathbin{\hat{\otimes}} Z \cong \int^{\alpha,\beta,\gamma:\mathbf{P}} X(\alpha) \times Y(\beta) \times Z(\gamma) \times \mathbf{P}(\alpha \otimes \beta \otimes \gamma, —)$$

■

Theorem 4.2 (Soundness)

1. *If* $\Gamma; \Omega \vdash R$ *then* $\Gamma; \Omega \models R$
2. *If* $\Gamma \vdash [P]$ *then* $\Gamma \models [P]$
3. *If* $\Gamma[N] \vdash Q$ *then* $\Gamma[N] \models Q$

Proof By induction on the derivation. Some example cases:

Case:

$$\frac{\Gamma; P \vdash Q}{\Gamma[\uparrow P] \vdash Q}$$

We assume a natural transformation

$$\zeta : (\llbracket \Gamma \rrbracket \mathbin{\hat{\multimap}} \llbracket Q \rrbracket^{>}) \mathrel{\dot{\to}} \llbracket \uparrow P \rrbracket$$

and need to produce a natural transformation

$$\xi : (\llbracket \Gamma \rrbracket \mathbin{\hat{\otimes}} \llbracket P \rrbracket) \wedge \llbracket Q \rrbracket^{>} \mathrel{\dot{\to}} \triangleright$$

Expanding definitions, we have

$$\zeta_\psi : \left(\int^{\alpha:\mathbf{P}, \phi:\mathbf{N}} \llbracket \Gamma \rrbracket(\alpha) \times \llbracket Q \rrbracket^{>}(\phi) \times \mathbf{N}(\alpha \multimap \phi, \psi) \right) \mathrel{\dot{\to}}$$

$$\int_{\beta:\mathbf{P}} \llbracket P \rrbracket(\beta) \to (\beta \triangleright \psi)$$

and we need

$$\xi_{\alpha,\phi} : \left(\int^{\alpha_1:\mathbf{P}, \alpha_2:\mathbf{P}} \llbracket \Gamma \rrbracket(\alpha_1) \times \llbracket P \rrbracket(\alpha_2) \times \mathbf{P}(\alpha_1 \otimes \alpha_2, \alpha) \right) \times \llbracket Q \rrbracket^{>}(\phi) \to (\alpha \triangleright \phi)$$

So let

$$\gamma : \llbracket \Gamma \rrbracket(\alpha_1) \qquad p : \llbracket P \rrbracket(\alpha_2) \qquad g : \mathbf{P}(\alpha_1 \otimes \alpha_2, \alpha) \qquad q : \llbracket Q \rrbracket^{>}(\phi)$$

be given. Then we have

$$\zeta_{\alpha_1 \multimap \phi}(\gamma, q, \mathsf{id}_{\alpha_1 \multimap \phi})(p) : \alpha_2 \triangleright (\alpha_1 \multimap \phi)$$

By k, this is as good as $(\alpha_1 \otimes \alpha_2) \triangleright \phi$, and we apply functoriality of $\triangleright$ to g and we're done.

Case:

$$\frac{\Gamma \vdash [P]}{\Gamma \vdash P}$$

We assume a natural transformation $\zeta : \llbracket \Gamma \rrbracket \mathrel{\dot{\to}} \llbracket P \rrbracket : \mathbf{P} \to \mathbf{Set}$, and need to produce a natural transformation $\xi : \llbracket \Gamma \rrbracket \wedge \llbracket \uparrow P \rrbracket \mathrel{\dot{\to}} \triangleright : \mathbf{P} \times \mathbf{N} \to \mathbf{Set}$.

For an object (α, ϕ) of $\mathbf{P} \times \mathbf{N}$, the arrow $\xi_{(\alpha,\phi)}$ must be a function

$$[\![\Gamma]\!](\alpha) \times \left(\int_{\alpha:\mathbf{P}} [\![P]\!](\alpha) \to (\alpha \triangleright \phi) \right) \to (\alpha \triangleright \phi)$$

Note that for any object $\alpha : \mathbf{P}$ there is a projection function out of the end

$$\pi_\alpha : \left(\int_{\alpha:\mathbf{P}} [\![P]\!](\alpha) \to (\alpha \triangleright \phi) \right) \to [\![P]\!](\alpha) \to (\alpha \triangleright \phi)$$

So we set

$$\xi_{(\alpha,\phi)} = \lambda(\gamma, e).\pi_\alpha(e)(\zeta_\alpha(\gamma))$$

Case:

$$\frac{\Gamma_1 \vdash [P_1] \qquad \Gamma_2 \vdash [P_1]}{\Gamma_1, \Gamma_2 \vdash [P_1 \otimes P_2]}$$

We assume natural transformations $\zeta_1 : [\![\Gamma_1]\!] \dot{\to} [\![P_1]\!]$ and $\zeta_2 : [\![\Gamma_2]\!] \dot{\to} [\![P_2]\!]$, and we need to produce

$$\xi : [\![\Gamma_1]\!] \hat{\otimes} [\![\Gamma_2]\!] \dot{\to} [\![P_1 \otimes P_2]\!]$$

In other words

$$\xi_\beta : \int^{\alpha_1,\alpha_2:\mathbf{P}} [\![\Gamma_1]\!](\alpha_1) \times [\![\Gamma_2]\!](\alpha_2) \times \mathbf{P}(\alpha_1 \otimes \alpha_2, \beta) \dot{\to}$$

$$\int^{\alpha_1,\alpha_2:\mathbf{P}} [\![P_1]\!](\alpha_1) \times [\![P_2]\!](\alpha_2) \times \mathbf{P}(\alpha_1 \otimes \alpha_2, \beta)$$

This is easily constructed out of ζ_1 and ζ_2 and $\mathsf{id}_{\mathbf{P}(-\otimes-,\beta)}$.

Case:

$$\frac{\Gamma; P_1, P_2 \vdash Q}{\Gamma; P_1 \otimes P_2 \vdash Q}$$

We have a natural transformation

$$[\![\Gamma]\!] \hat{\otimes} [\![P_1]\!] \hat{\otimes} [\![P_2]\!] \hat{\otimes} [\![\uparrow Q]\!] \dot{\to} \triangleright$$

and need to produce

$$[\![\Gamma]\!] \hat{\otimes} [\![P_1 \otimes P_2]\!] \hat{\otimes} [\![\uparrow Q]\!] \dot{\to} \triangleright$$

which is essentially the same thing, by associativity of $\hat{\otimes}$.

■

4.1 Completeness

We build a syntactic model. Fix a collection of atoms.

- Let **P** be the category whose objects are all contexts Γ, and whose morphisms are permutations. The functor $\otimes$ is context concatenation, and the monoidal unit I is the empty context.
- Let **N** be the category whose objects are pairs (Γ, Q) and whose morphisms are context permutations. The functor $\multimap$ takes an object Γ_1 of **P** and an object (Γ_2, Q) of **N** and produces $((\Gamma_1, \Gamma_2), Q)$ in **N**.
- The functor $\triangleright$ takes an object Γ_1 of **P** and an object (Γ_2, Q) of **N** and yields the set of derivations $\Gamma_1, \Gamma_2 \vdash Q$.
- k is easy to check by simply constructing proofs and using cut.
- We choose as interpretations of the atoms as follows

$$\eta(a^+)(\Gamma) = \begin{cases} \{*\} & \text{if } \Gamma = (\langle a^+ \rangle); \\ \emptyset & \text{otherwise.} \end{cases}$$

$$\eta(a^-)(\Gamma, Q) = \begin{cases} \{*\} & \text{if } (\Gamma, Q) = (\cdot, \langle a^- \rangle); \\ \emptyset & \text{otherwise.} \end{cases}$$

The thing we need to do is show that this model is in fact universal — that the interpretation of every proposition reflects its (focused) provability. We first claim that

Lemma 4.3 *In the syntactic model, for all P and N we have*

- *If $[\![P]\!](\Gamma)$, then $\Gamma \vdash [P]$*
- *If $[\![N]\!](\Gamma, Q)$, then $\Gamma[N] \vdash Q$*

Proof This proceeds by induction on the proposition. For the atoms, the definition of η makes this immediately true.

Note that because the categories **P** and **N** are groupoids, taking (co)ends over them is the same as taking mere (co)products over their connected components, for example

$$\int^{\Gamma:\mathbf{P}} F(\Gamma, \Gamma) = \coprod_{\Gamma:\mathbf{P}} F(\Gamma, \Gamma)$$

where the $\coprod$ is understood as ranging over contexts identified up to permutation. Other cases:

Case: $P_1 \otimes P_2$. If $[\![P_1 \otimes P_2]\!](\Gamma)$ is inhabited, then by definition

$$\coprod_{\Gamma_1, \Gamma_2 : \mathbf{P}} [\![P_1]\!](\Gamma_1) \times [\![P_1]\!](\Gamma_2) \times \mathbf{P}((\Gamma_1, \Gamma_2), \Gamma)$$

is inhabited. By i.h., we have $\Gamma_1 \vdash [P_1]$ and $\Gamma_2 \vdash [P_2]$, so $\Gamma_1, \Gamma_2 \vdash [P_1 \otimes P_2]$ as required.

Case: $\downarrow N$. Suppose $[\![\downarrow N]\!](\Gamma)$ is inhabited. By definition,

$$\prod_{(\Gamma_0,Q):\mathbf{N}} [\![N]\!](\Gamma_0, Q) \to \Gamma \rhd (\Gamma_0, Q)$$

is inhabited. By Lemma 2.2 and soundness, we have $\Gamma \vdash N$.

■

Lemma 4.4 *In the syntactic model,*

1. $[\![\downarrow N]\!](N)$ *is inhabited.*
2. $[\![\uparrow P]\!](\cdot, P)$ *is inhabited.*
3. $[\![\Gamma]\!](\Gamma)$ *is inhabited.*
4. $([\![\Gamma]\!] \hat{\multimap} [\![Q]\!]^{>})(\Gamma, Q)$ *is inhabited.*

Proof Follows easily from Lemma 2.2. ■

Lemma 4.5 *In the syntactic model,*

1. *If* $([\![\Gamma]\!] \hat{\otimes} [\![\Omega]\!]) \wedge [\![R]\!]^{>} \dot{\to} \rhd$, *then* $\Gamma; \Omega \vdash R$.
2. *If* $[\![\Gamma]\!] \dot{\to} [\![P]\!]$, *then* $\Gamma \vdash [P]$.
3. *If* $([\![\Gamma]\!] \hat{\multimap} [\![Q]\!]^{>}) \dot{\to} [\![N]\!]$, *then* $\Gamma[N] \vdash Q$.

Proof Combine Lemmas 4.4 and 4.3. ■

Corollary 4.6 (Completeness)

1. *If* $\Gamma; \Omega \models R$ *then* $\Gamma; \Omega \vdash R$
2. *If* $\Gamma \models [P]$ *then* $\Gamma \vdash [P]$
3. *If* $\Gamma[N] \models Q$ *then* $\Gamma[N] \vdash Q$

References

[HS07] Masahiro Hamano and Philip Scott. A categorical semantics for polarized MALL. *Annals of Pure and Applied Logic*, 145(3):276 – 313, 2007.