# Priorities and Comonads

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**Definition** A *priority* on a category **C** with finite limits is a pair $(P, \sigma)$ where $P$ is a lex endofunctor on **C** and $\sigma$ is a natural transformation $P \to P^2$ such that the diagram

$$\begin{tikzcd}
P^3 & P^2 \arrow[l, "\sigma_P"'] \\
P^2 \arrow[u, "P\sigma"] & P \arrow[l, "\sigma"] \arrow[u, "\sigma"']
\end{tikzcd} \qquad (i)$$

commutes, and the diagram

$$\begin{tikzcd}
P^2(X \times Y) & P(X \times Y) \arrow[l, "\sigma_{X \times Y}"'] \\
P(PX \times PY) \arrow[u, "Pq_{XY}"] & \\
P^2X \times P^2Y \arrow[u, "q_{PX,PY}"] & PX \times PY \arrow[l, "\sigma_X \times \sigma_Y"] \arrow[uu, "q_{XY}"']
\end{tikzcd} \qquad (ii)$$

commutes for any objects $X, Y$ of **C**, where $q_{XY}$ is the canonical natural isomorphism $PX \times PY \to P(X \times Y)$ induced by $P$ being lex.

**Lemma 0.1** *Let $(P, \sigma)$ be a priority on a category* **C**. *Then the diagram*

$$\begin{tikzcd}
PA \arrow[r, "\sigma_A"] \arrow[d, "{(\sigma_A, 1_{PA})}"'] & P^2A \arrow[r, "{P(\sigma_A, 1_{PA})}"] & PTPA \arrow[d, "Pq_{PA,A}"] \\
TPA \arrow[r, "q_{PA,A}"'] & PTA \arrow[r, "\sigma_{TA}"'] & P^2TA
\end{tikzcd}$$

*commutes.*

**Proof** We first observe that

$$
\begin{array}{rcl}
(P\mathrm{fst}_{TPA})P(\sigma_A, 1_{PA})\sigma_A & = & P(\mathrm{fst}_{TPA}(\sigma_A, 1_{PA}))\sigma_A \\
& = & P\sigma_A\sigma_A \\
& = & P\sigma_A\mathrm{fst}_{TPA}(\sigma_A, 1_{PA}) \\
& = & \mathrm{fst}_{TP^2A}(P\sigma_A \times \sigma_A)(\sigma_A, 1_{PA}) \\
& = & (P\mathrm{fst}_{TPA})q_{P^2A,PA}(P\sigma_A \times \sigma_A)(\sigma_A, 1_{PA})
\end{array}
$$

and

$$
\begin{array}{rcl}
(P\mathrm{snd}_{TPA})P(\sigma_A, 1_{PA})\sigma_A & = & P(\mathrm{snd}_{TPA}(\sigma_A, 1_{PA}))\sigma_A \\
& = & \sigma_A \\
& = & \sigma_A\mathrm{snd}_{TPA}(\sigma_A, 1_{PA}) \\
& = & \mathrm{snd}_{TP^2A}(P\sigma_A \times \sigma_A)(\sigma_A, 1_{PA}) \\
& = & (P\mathrm{snd}_{TPA})q_{P^2A,PA}(P\sigma_A \times \sigma_A)(\sigma_A, 1_{PA})
\end{array}
$$

hence (by pairing the two equations) if we set $h := (P\mathrm{fst}_{TPA}, P\mathrm{snd}_{TPA})$ we have $hP(\sigma_A, 1_{PA})\sigma_A = hq_{P^2A,PA}(P\sigma_A \times \sigma_A)(\sigma_A, 1_{PA})$. But it is easily seen that $h = q^{-1}$, so $h$ is mono and therefore $P(\sigma_A, 1_{PA})\sigma_A = q_{P^2A,PA}(P\sigma_A \times \sigma_A)(\sigma_A, 1_{PA})$. Now if we take the diagram $(i)$ and multiply it with the (trivially commutative) diagram

$$
\begin{tikzcd}
PA \arrow[r, "1_{PA}"] \arrow[d, "1_{PA}"'] & PA \arrow[d, "\sigma_A"] \\
PA \arrow[r, "\sigma_A"'] & P^2A
\end{tikzcd}
$$

we obtain

$$
\begin{tikzcd}
PA \times PA \arrow[r, "\sigma_A \times PA"] \arrow[d, "\sigma_A \times PA"'] & TPA \arrow[d, "\sigma_{PA} \times \sigma_A"] \\
TPA \arrow[r, "P\sigma_A \times \sigma_A"'] & TP^2A
\end{tikzcd}
$$

and since $(\sigma_A \times PA)\Delta_{PA} = (\sigma_A, 1_{PA})$, we have that the diagram

$$\begin{tikzcd}
PA \arrow[r, "{(\sigma_A, PA)}"] \arrow[d, "{(\sigma_A, PA)}"'] & TPA \arrow[d, "\sigma_{PA} \times \sigma_A"] \\
TPA \arrow[r, "P\sigma_A \times \sigma_A"'] & TP^2A
\end{tikzcd}$$

commutes. Putting all of the above together with diagram $(ii)$ with $X := PA$ and $Y := A$, we obtain the commutativity of

$$\begin{tikzcd}
PA \arrow[rr, "\sigma_A"] \arrow[dr, "{(\sigma_A, 1_{PA})}"] \arrow[dd, "{(\sigma_A, 1_{PA})}"'] & & P^2A \arrow[dd, "{P(\sigma_A, 1_{PA})}"] \\
& TPA \arrow[d, "P\sigma_A \times \sigma_A"'] & \\
TPA \arrow[r, "\sigma_{PA} \times \sigma_A"'] \arrow[d, "q_{PA,A}"'] & TP^2A \arrow[r, "q_{P^2A,PA}"'] & PTPA \arrow[d, "Pq_{PA,A}"] \\
PTA \arrow[rr, "\sigma_{TA}"'] & & P^2TA
\end{tikzcd}$$

as required. ■

**Theorem 0.2** *Let $(P, \sigma)$ be a priority on a category $\mathbf{C}$. If we make the definitions*

$$\begin{aligned}
TA &:= PA \times A \\
\epsilon_A &:= \mathrm{snd}_{PA \times A} \\
\delta_A &:= (q_{PA,A}(\sigma_A, 1_{PA}) \times (PA \times A))a_{PA,PA,A}(\Delta_{PA} \times A)
\end{aligned}$$

*(where $\Delta_A := (1_A, 1_A)$ and $a_{ABC}$ is the canonical natural isomorphism $(A \times B) \times C \to A \times (B \times C)$) then $(T, \epsilon, \delta)$ is a comonad.*

**Proof** We easily have that $\epsilon$ and $\delta$ are natural because 'everything in sight' is natural in their definition. To simplify future computations, we define

$$\alpha_A \quad := \quad (q_{PA,A}(\sigma_A, 1_{PA}) \times (PA \times A))$$

$$\begin{array}{rcl}\beta_A & := & a_{PA,PA,A} \\ \gamma_A & := & \Delta_{PA} \times A\end{array}$$

so that $\delta = \alpha\beta\gamma$. It remains to show that the comonad equalities are satisfied. We see first that the diagram

$$\begin{tikzcd} TA & T^2A \arrow[l, "\mathrm{snd}_{T^2A}"'] \arrow[r, "T\mathrm{snd}_{TA}"] & TA \\ & TA \arrow[ul, "1_{TA}"] \arrow[u, "\delta_A"] \arrow[ur, "1_{TA}"'] & \end{tikzcd} \qquad (*)$$

commutes, because

$$\begin{array}{rcl}\mathrm{snd}_{T^2A}\delta_A & = & \mathrm{snd}_{T^2A}\alpha_A\beta_A\gamma_A \\ & = & \mathrm{snd}_{PA\times(PA\times A)}\beta_A\gamma_A \\ & = & (\mathrm{fst}_{PA\times A}\mathrm{snd}_{PA\times(PA\times A)}, \mathrm{snd}_{PA\times A}\mathrm{snd}_{PA\times(PA\times A)})\beta_A\gamma_A \\ & = & (\mathrm{fst}_{PA\times A}\mathrm{snd}_{PA\times(PA\times A)}\beta_A, \mathrm{snd}_{PA\times A}\mathrm{snd}_{PA\times(PA\times A)}\beta_A)\gamma_A \\ & = & (\mathrm{snd}_{PA\times PA}\mathrm{fst}_{(PA\times PA)\times A}, \mathrm{snd}_{(PA\times PA)\times A})\gamma_A \\ & = & (\mathrm{snd}_{PA\times PA}\mathrm{fst}_{(PA\times PA)\times A}\gamma_A, \mathrm{snd}_{(PA\times PA)\times A}\gamma_A) \\ & = & (\mathrm{snd}_{PA\times PA}\Delta_{PA}\mathrm{fst}_{PA\times A}, \mathrm{snd}_{PA\times A}) \\ & = & (\mathrm{fst}_{PA\times A}, \mathrm{snd}_{PA\times A}) \\ & = & 1_{PA\times A}\end{array}$$

and

$$\begin{array}{rcl}(T\epsilon_A)\delta_A & = & T\mathrm{snd}_{TA}\alpha_A\beta_A\gamma_A \\ & = & (P\mathrm{snd}_{TA} \times \mathrm{snd}_{TA})\alpha_A\beta_A\gamma_A \\ & = & (P\mathrm{snd}_{TA}q_{PA,A}(\sigma_A, 1_{PA}) \times \mathrm{snd}_{TA})\beta_A\gamma_A \\ & = & (\mathrm{snd}_{P^2A\times PA}(\sigma_A, 1_{PA}) \times \mathrm{snd}_{TA})\beta_A\gamma_A \\ & = & (1_{PA} \times \mathrm{snd}_{TA})\beta_A\gamma_A \\ & = & (\mathrm{fst}_{PA\times(PA\times A)}, \mathrm{snd}_{PA\times A}\mathrm{snd}_{PA\times(PA\times A)})\beta_A\gamma_A \\ & = & (\mathrm{fst}_{PA\times(PA\times A)}\beta_A, \mathrm{snd}_{PA\times A}\mathrm{snd}_{PA\times(PA\times A)}\beta_A)\gamma_A \\ & = & (\mathrm{fst}_{PA\times PA}\mathrm{fst}_{(PA\times PA)\times A}, \mathrm{snd}_{(PA\times PA)\times A})\gamma_A \\ & = & (\mathrm{fst}_{PA\times PA}\mathrm{fst}_{(PA\times PA)\times A}\gamma_A, \mathrm{snd}_{(PA\times PA)\times A}\gamma_A) \\ & = & (\mathrm{fst}_{PA\times PA}\Delta_{PA}\mathrm{fst}_{PA\times A}, \mathrm{snd}_{PA\times A}) \\ & = & (\mathrm{fst}_{PA\times A}, \mathrm{snd}_{PA\times A}) \\ & = & 1_{PA\times A}\end{array}$$

This immediately provides that $\epsilon$ behaves correctly as a counit. Now we want to show that $\delta_T\delta = (T\delta)\delta$. If we hit the left triangle of $(*)$ 'on the inside' with

$T$, we see that the following diagram commutes:

$$\begin{tikzcd}
T^2A & T^3A \arrow[l, "\mathrm{snd}_{T^3A}"'] \\
& T^2A \arrow[ul, "1_{T^2A}"] \arrow[u, "\delta_{TA}"']
\end{tikzcd} \qquad (**)$$

We can then verify that $\mathrm{snd}_{T^3}\delta_{TA}\delta_A = \mathrm{snd}_{T^3}(T\delta_A)\delta_A$ by chasing the diagram resulting from pasting together $(**)$ with the left triangle of $(*)$ and the naturality of snd and $1_{\mathbf{C}}$:

$$\begin{tikzcd}
& TA \arrow[dl, "\delta_A"'] \arrow[d, "1_{TA}"] \arrow[dr, "\delta_A"] & \\
T^2A \arrow[d, "\delta_{TA}"'] \arrow[dr, "1_{T^2A}"] & TA \arrow[d, "\delta_A"] & T^2A \arrow[l, "\mathrm{snd}_{T^2A}"] \arrow[d, "T(\delta_A)"] \\
T^3A \arrow[r, "\mathrm{snd}_{T^3A}"'] & T^2A & T^3A \arrow[l, "\mathrm{snd}_{T^3A}"]
\end{tikzcd}$$

Next we observe that $\mathrm{fst}_{T^2A}\delta_A = q_{PA,A}(\sigma_A, 1_{PA})\mathrm{fst}_{TA}$, for the following diagram commutes:

$$\begin{tikzcd}
TA \arrow[rrr, "\delta_A"] \arrow[dr, "\gamma_A"] \arrow[ddd, "\mathrm{fst}_{TA}"'] & & & T^2A \arrow[dd, "\mathrm{fst}_{T^2A}"] \\
& (PA)^2 \times A \arrow[r, "\beta_A"] \arrow[d, "\mathrm{fst}_{(PA)^2\times A}"] & PA \times TA \arrow[ur, "\alpha_A"] \arrow[dd, "\mathrm{fst}_{PA\times TA}"] & \\
& (PA)^2 \arrow[dr, "\mathrm{fst}_{(PA)^2}"] & & PTA \\
PA \arrow[ur, "\Delta_{PA}"] \arrow[rr, "1_{PA}"'] & & PA \arrow[r, "{(\sigma_A, 1_{PA})}"'] & TPA \arrow[u, "q_{PA,A}"']
\end{tikzcd}$$

from this we can conclude that the diagram

$$
\begin{tikzcd}
T^3 \arrow[rr, "\mathrm{fst}_{T^3A}"] & & PT^2A \arrow[dd, "P\mathrm{snd}_{T^2A}"] \\
T^2A \arrow[u, "T\delta_A"] \arrow[r, "\mathrm{fst}_{T^2A}"] \arrow[dd, "\delta_{TA}"'] & PTA \arrow[ur, "P\delta_A"] \arrow[d, "(\sigma_{TA}, 1_{PTA})"'] \arrow[dr, "1_{PTA}"] & \\
& TPTA \arrow[r, "\mathrm{snd}_{TPTA}"'] \arrow[dr, "q_{P^2A,PA}"'] & PTA \\
T^3A \arrow[rr, "\mathrm{fst}_{T^3A}"'] & & PT^2A \arrow[u, "P\mathrm{snd}_{T^2A}"']
\end{tikzcd}
$$

commutes. So we know

$$\mathrm{snd}\delta_T\delta = \mathrm{snd}(T\delta)\delta$$

and

$$(P\mathrm{snd})\mathrm{fst}\delta_T\delta = (P\mathrm{snd})\mathrm{fst}(T\delta)\delta$$

If we can show $(P\mathrm{fst})\mathrm{fst}\delta_T\delta = (P\mathrm{fst})\mathrm{fst}(T\delta)\delta$, then it follows that $\delta_T\delta = (T\delta)\delta$ and we are done. To see this, observe that the following two diagrams are commutative, and apply Lemma 0.1.

$$
\begin{tikzcd}
TA \arrow[rr, "\delta_A"] \arrow[dd, "\mathrm{fst}_{TA}"'] & & T^2A \arrow[r, "T\delta_A"] \arrow[d, "\mathrm{fst}_{T^2A}"] & T^3A \arrow[d, "\mathrm{fst}_{T^3A}"] \\
& TPA \arrow[r, "q_{PA,A}"] \arrow[d, "\mathrm{fst}_{TPA}"] & PTA \arrow[r, "P\delta_A"'] \arrow[dl, "P\mathrm{fst}_{TA}"'] & PT^2A \arrow[d, "P\mathrm{fst}_{T^2A}"] \\
PA \arrow[r, "\sigma_A"'] \arrow[ur, "(\sigma_A, 1_{PA})"] & P^2A \arrow[r, "P(\sigma_A, 1_{PA})"'] & PTPA \arrow[r, "Pq_{PA,A}"'] & P^2TA
\end{tikzcd}
$$

$$
\begin{tikzcd}
TPA \arrow[r, "q_{PA,A}"] & PTA \arrow[r, "\sigma_{TA}"] \arrow[dr, "{(\sigma_{TA}, 1_{PTA})}"] & P^2TA & \\
PA \arrow[u, "{(\sigma_A, 1_{PA})}"] & & TPTA \arrow[u, "\mathrm{fst}_{TPTA}"] \arrow[r, "q_{PTA,TA}"'] & PT^2A \arrow[ul, "P\mathrm{fst}_{T^2A}"'] \\
TA \arrow[u, "\mathrm{fst}_{TA}"] \arrow[r, "\delta_A"'] & T^2A \arrow[uu, "\mathrm{fst}_{T^2A}"] \arrow[rr, "\delta_{TA}"'] & & T^3A \arrow[u, "\mathrm{fst}_{T^3A}"']
\end{tikzcd}
$$

■