Focus-preserving Embeddings of Substructural Logics in Intuitionistic Logic

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Abstract

We present a method of embedding substructural logics into ordinary first-order intuitionistic logic. This embedding is faithful in a very strong sense: not only does it preserve provability of sequents under translation, but it also preserves sets of proofs — and the focusing structure of those proofs — up to isomorphism. Examples are given for the cases of intuitionistic linear logic and ordered logic, and indeed we can use our method to derive a correct focusing system for ordered logic. Potential applications lie in logic programming, theorem proving, and logical frameworks for substructural logics where focusing is crucial for the underlying proof theory.

1 Introduction

Substructural logics can enforce restrictions on use of hypotheses by having a structured context: hypotheses in linear logic [Gir87], where the context is a multiset, must be used exactly once, and in ordered logic [Lam58, Pol01] are used in a specified order, because the context is a list. In the same vein as display logic [Bel82] and work on graphical representations of structured contexts [Lam07], we show how diverse substructural logics can be treated uniformly by isolating the reasoning about the algebraic properties of their context's structure. Unlike these other approaches, we do so without introducing a logic that itself has a sophisticated notion of structured context, and instead use *focused* proofs [And92] in a very simple *nonsubstructural* logic. This reduction of substructural to nonsubstructural has proved useful in understanding the design of substructural (especially dependent) type theories. We specifically show how to embed substructural logics into in a fragment of focused first-order intuitionistic logic with equality, over a signature of function symbols suited to the substructural logic being embedded.

The embedding can be viewed as a constructive resource interpretation of substructural logics, where the first-order domain provides the notion of resources. Because the proof system of the target language is limited, and compatible with focusing, we are able to formulate and prove much stronger claims about the faithfulness of our embedding than can usually be obtained for standard resource semantics into classical algebraic structures. Not only does provability coincide with provability across the embedding, but proofs correspond bijectively to proofs, and focusing phases to focusing phases.

Focusing is deeply connected to notions of uniform proof for logic programming, to the analysis of canonical forms in dependently-typed logical frameworks, and to efficient automated proof search procedures. By showing how to relocate the problem of understanding focusing systems of substructural logics to the setting of a simpler and more easily understood calculus, we open the door to more (and more convenient) application of the expressivity of substructural logics in all of these areas.

The rest of the paper is organized toward the aim of treating two examples of the embedding, intuitionistic linear logic (Section 3) and ordered logic (Section 4), but we must first describe the representation language into which they will be embedded (Section 2). We finish with a discussion of related work (Section 5) and conclusions about our contribution (Section 6).

2 The Logic of FF

Our representation language is FF, for **F**ocused **F**irst-order intuitionistic logic. We refer in the sequel to the substructural logic being embedded as the *object language*.

The notion of *focusing*, introduced by Andreoli [And92], is a way of narrowing eligible proofs down to those that decompose connectives in maximal contiguous runs of logical connectives of the same *polarity*. Polarity is a trait of propositional connectives which, among other properties, characterizes whether they can be eagerly decomposed as goals (negative propositions) or eagerly decomposed as assumptions (positive propositions). Importantly, focused proof search is *complete*: there is a focused proof of a proposition iff there is an ordinary proof, but there are generally fewer *distinct* focused proofs. It is by using the tight control over proof search and proof identity that focusing affords that we are able to faithfully mimic not only of *which* propositions are provable the object language, but *how* they are proved.

FF is a multi-sorted first-order logic, and is parametrized over the structure of its first-order domain: we leave it open for each particular embedding to choose a collection of sorts, which function symbols exist to build terms of those sorts, and how an equivalence relation $\equiv$ on those terms is axiomatized.

In the embedding, first-order terms serve to describe the shape of sequents with substructural contexts. The relation $\equiv$ is used to express that two sequent shapes are considered equivalent. For instance, in the case linear logic, it captures the property that the order of hypotheses does not matter.

2.1 Syntax

The basic syntax of FF is as follows.

$$
\begin{array}{rl}
\text{Negative Props } A ::= & B \Rightarrow A \mid A \wedge A \mid \top \mid \forall x{:}\sigma.A \mid s^- \\
\text{Positive Props } B ::= & s^+ \mid {\downarrow}A \\
\text{Contexts } \Gamma ::= & \cdot \mid \Gamma, B \\
\text{Sorts } \sigma ::= & \cdots \\
\text{Terms } t ::= & x \mid \cdots
\end{array}
$$

The bulk of the propositions are built of *negative* logical connectives: implication, conjunction, truth, universal quantification, and negative atomic propositions s^-. Atomic propositions, both positive and negative (hereafter sometimes 'atoms'), are generally predicates on first-order terms. We leave it again to the particular embedding to decide which atoms (i.e. which predicates) exist in the language, and of which polarity. The first argument to implication is, as usual for focusing systems, a *positive* proposition. Ordinarily the positives might include existential quantification and disjunction, but for our purposes we only need positive atoms s^+, and an inclusion of negative propositions back into positives, via the shift connective $\downarrow$ that interrupts focus phases.

Contexts Γ are built out of positive propositions, and are subject to tacit weakening, contraction, and exchange.

The judgments of the system are:

Right Focus	$\Gamma \vdash [B]$
Left Focus	$\Gamma; [A] \vdash s^-$
Right Inversion	$\Gamma \vdash A$

We will also use the defined judgment of equivalence $s_1^- \equiv s_2^-$ of negative atoms, which is defined to mean that s_1^- and s_2^- have the same predicate symbol, and each of the corresponding pairs of term arguments are related by $\equiv$.

The focus judgments are used when we have selected a proposition and have committed to continue decomposing it until we reach a polarity shift. Inversion takes place when we are trying to prove a negative proposition, and we apply right rules eagerly, because all right rules for negative propositions are characteristically invertible.

For uniformity, we write $\Gamma \vdash J$ to stand for either $\Gamma; [A] \vdash s^-$ or $\Gamma \vdash A$. On occasion, when we need to contrast the judgment of FF with that of the object language, decorate the turnstile as $\vdash_{\text{FF}}$.

2.2 Proof Theory

The valid deductions of this judgment are defined by the inference rules in Figure 1. They are mostly standard, but we note some consequences of focusing discipline: when we are focused on a negative atomic proposition s^-, the current conclusion s_0^- must be already equivalent to s^-; when focused on a positive atom s^+, that same atom must already be found in the current context. Encountering ${\downarrow}A$ on the right blurs focus, and begins inversion of A. Decomposing $\downarrow$ on the left begins a focus phase, which is only allowed once the conclusion has finished inversion, and arrived at a negative atomic proposition s^-.

The right rule for the quantifier is understood to have the usual side conditions about the freshness of variable it introduces. We write $\{t/x\}$ for substitution of a first-order term for the variable x. By $\vdash t : \sigma$ we mean that t is a well-formed term of sort σ; for space reasons we avoid giving a complete explanation of this rather standard notion. We will write $\mathsf{f} : (\sigma_1, \ldots, \sigma_n) \to \sigma$ to indicate that f is a function symbol taking n arguments of sorts $\sigma_1, \ldots, \sigma_n$ and yielding a term of sort σ.

2.3 Metatheory

This calculus satisfies the usual pair of properties that establish its internal soundness (cut admissibility) and internal completeness (identity expansion). Because of the equivalence relation allowed at negative atoms, we must first show a congruence lemma with respect to the relation.

Lemma 2.1 (Congruence) *Suppose $s^- \equiv s_0^-$.*

1. *If $\Gamma \vdash s^-$, then $\Gamma \vdash s_0^-$*
2. *If $\Gamma; [A] \vdash s^-$, then $\Gamma; [A] \vdash s_0^-$*

Proof By induction on the derivation. Use transitivity for the case of $s^- L$. ■

The admissibility of cut now follows.

Theorem 2.2 (Cut Admissibility) *The following rules are admissible:*

$$
\frac{\Gamma \vdash [B] \qquad \Gamma, B \vdash J}{\Gamma \vdash J}
\qquad
\frac{\Gamma \vdash A \qquad \Gamma, {\downarrow}A \vdash J}{\Gamma \vdash J}
$$

$$\frac{}{\Gamma, s^+ \vdash [s^+]}\, s^+R \qquad \frac{s_1^- \equiv s_2^-}{\Gamma; [s_1^-] \vdash s_2^-}\, s^-L \qquad \frac{\Gamma \vdash A}{\Gamma \vdash [\downarrow A]}\, {\downarrow}R \qquad \frac{\Gamma, \downarrow A; [A] \vdash s^-}{\Gamma, \downarrow A \vdash s^-}\, {\downarrow}L \qquad \frac{\Gamma \vdash A_1 \qquad \Gamma \vdash A_2}{\Gamma \vdash A_1 \wedge A_2}\, {\wedge}R \qquad \frac{\Gamma; [A_i] \vdash s^-}{\Gamma; [A_1 \wedge A_2] \vdash s^-}\, {\wedge}L_i$$

$$\frac{}{\Gamma \vdash \top}\, \top R \qquad \frac{\Gamma, B \vdash A}{\Gamma \vdash B \Rightarrow A}\, {\Rightarrow}R \qquad \frac{\Gamma \vdash [B] \qquad \Gamma; [A] \vdash s^-}{\Gamma; [B \Rightarrow A] \vdash s^-}\, {\Rightarrow}L \qquad \frac{\Gamma \vdash A}{\Gamma \vdash \forall x{:}\sigma.A}\, \forall R^x \qquad \frac{\vdash t : \sigma \qquad \Gamma; [\{t/x\}A] \vdash s^-}{\Gamma; [\forall x{:}\sigma.A] \vdash s^-}\, \forall L$$

Figure 1. FF Inference Rules

$$\frac{\Gamma \vdash A \qquad \Gamma; [A] \vdash s^-}{\Gamma \vdash s^-}$$

Proof By a standard structural cut admissibility proof, using lexicographic induction on the cut formula B or A and the derivations involved. In the first rule, if B is an atom we are done, by the admissibility of contraction. Otherwise analyze the second premise. In the second rule, split cases on the premise $\Gamma \vdash A$. In the third rule, both premises are analyzed in lockstep; when A is a negative atom, use the above congruence lemma. ∎

We can also obtain the result that shows every proposition (not just any atomic proposition) entails itself:

Theorem 2.3 (Identity Expansion) *For all* Γ, A, B, *we have* $\Gamma, \downarrow A \vdash A$ *and* $\Gamma, B \vdash [B]$.

Proof See appendix.

3 Embedding Linear Logic

In this section we show how to embed focused intuitionistic linear logic into FF in a proof-preserving way. We will then obtain that ordinary unfocused linear logic can also be embedded; we need only apply the usual insertion of shift connectives in a way that focused proof search on the result simulates proof search on the original proposition.

Although embeddings of classical calculi into FF are also certainly possible, all of the examples herein are embeddings of intuitionistic systems, and we refer simply to 'linear logic' throughout and understand it to mean the intuitionistic variety.

The language of focused linear logic,[1] just as in FF, has polarized propositions, negative and positive, with polarity shift connectives $\uparrow$ and $\downarrow$ passing between them. Their syntax is as follows.

Negative N	::=	$\uparrow P \mid N \,\&\, N \mid \top \mid P \multimap N \mid a^-$
Positive P	::=	$\downarrow N \mid P \otimes P \mid 1 \mid P \oplus P \mid 0 \mid a^+ \mid\, !N$

Now to instantiate FF, we first choose a set of sorts, and function symbols to inhabit them. We will have three sorts,

$$\text{Sorts } \sigma ::= \mathsf{world} \mid \mathsf{frame} \mid \mathsf{struct}$$

[1] *[we should probably reference something here, like the work on polarized linear logic... -fp]*

to classify *worlds*, *frames*, and *structures*, with function symbols $* : (\mathsf{world}, \mathsf{world}) \to \mathsf{world}$, $\epsilon : () \to \mathsf{world}$, and $\lhd : (\mathsf{frame}, \mathsf{world}) \to \mathsf{struct}$. If we write the binary operations infix, and write α for world variables and ϕ for frame variables (we will have no need for structure variables) this can equivalently be seen as the declaration of syntax

Worlds p	::=	$\alpha \mid \epsilon \mid p * p$
Frames f	::=	$\phi \mid f \circledast p$
Structures r	::=	$f \lhd p$

Having made these conventions for variable names will let us elide sort declarations from uses of $\forall$ in the sequel. We must also choose what the atomic propositions are, and do so as follows: take negative atoms s^- to be exactly the *structures* $f \lhd p$, (or rather, posit a single one-place predicate on structures, but it is convenient to write it, as it were, with the empty predicate symbol) and positive atoms s^+ to be either $a^- @ f$ or $a^+ @ p$. These are just pairs, where one element is an *object-language* negative or positive atomic proposition a^- or a^+, and the other element is a frame or world, respectively.

Finally we must define an equivalence relation on first-order terms, and so we take as axioms the following

$$\epsilon * p \equiv p \qquad p * q \equiv q * p$$

$$p * (q * r) \equiv (p * q) * r \qquad (f \circledast p) \lhd q \equiv f \lhd (p * q)$$

plus symmetry, reflexivity, and transitivity of $\equiv$, and all congruence laws as expected; for example, $p * q \equiv p' * q'$ when $p \equiv p'$ and $q \equiv q'$.

Let us make a few comments to foreshadow the role of these three sorts of terms in the embedding. A world expression is used to represent the structure of a linear context. The empty world ϵ corresponds to the empty context, $*$ corresponds to the ability to take the multiset union of two contexts, and world variables α will label individual linear resources in the context. Since $*$ was made commutative and associative, and to satisfy unit laws with respect to ϵ, world expressions do behave like linear contexts.

The role of frames is less immediately intuitive: a frame represents the shape of a sequent with one hypothesis removed. In fact, this notion is in a sense dual to worlds, which are the shape of a sequent with one conclusion (of which there is only one!) removed. The need for frames arises from the need to represent decomposition (in the object language) of a positive hypothesis, such as $P_1 \otimes P_2$, that may leave behind a collection of new hypotheses in the context. We represent the act of choosing such a proposition to be decomposed by choice of a partition of the entire sequent into the proposition being focused on (a world variable) and the remainder of the sequent (a frame). This is particularly important in the case of ordered logic below, where the *location* of a hypothesis in the context matters — then the frame keeps track of which other hypotheses were to the left and to the right of the chosen proposition, and determine *where* the results of focus affect the context after decomposition.

The operation $\circledast$ builds up frame expressions, by adjoining more hypotheses to the context part of a frame. The above axiom for $\equiv$ involving $\circledast$ expresses a kind of associativity among $\circledast$, $\triangleleft$, $*$. Frame variables, dual to the way world variables label linear hypotheses, are used to label object language conclusions. Again, since our setting is intuitionistic, there is only one conclusion at a time, but it will still be important that different conclusions appearing in the same proof receive different labels.

Finally, a *structure* $f \triangleleft p$ represents the shape of an entire sequent, where we may imagine the context represented by p being substituted into the hole of the frame f.

3.1 Focused Linear Logic

We now describe the object language of focused linear logic [And92, LM09]. We follow *dual intuitionistic linear logic* [BP96] in having two separate contexts of *linear* hypotheses from a context of ordinary *unrestricted* hypotheses, a choice which goes back to Andreoli [And92] who explicitly relates a *dyadic system* of classical linear logic to a focused system for classical linear logic. Furthermore, for the inversion phases of focusing we need an ordered context of positive propositions to ensure that asynchronous (i.e. positive) hypotheses are decomposed in a deterministic order. The syntax of contexts and conclusions is:

Left Stable $\bar{N}$	$::=$	$N \mid a^+$
Right Stable $\bar{P}$	$::=$	$P \mid a^-$
Linear Contexts Δ	$::=$	$\cdot \mid \Delta, \bar{N}$
Unrest. Contexts Ψ	$::=$	$\cdot \mid \Psi, N$
Asynch. Contexts Ω	$::=$	$\cdot \mid \Omega, P$

Positive atoms a^+ and negative propositions N are both *stable* on the left since we can perform no more inversion on them there. We group them together and denote them as $\bar{N}$. They are precisely the propositions allowed in linear contexts. Conversely negative atoms a^- and positive propositions P, are stable on the right, as conclusions, and are denoted $\bar{P}$. Unrestricted contexts Ψ are subject to weakening, contraction, and exchange, linear contexts Δ to exchange, and asynchronous contexts Ω to no structural properties.

The judgments of focused linear logic are:

Stable	$\Psi; \Delta \vdash \bar{P}$
Right Focus	$\Psi; \Delta \vdash [P]$
Left Focus	$\Psi; \Delta; [N] \vdash \bar{P}$
Right Inversion	$\Psi; \Delta; \Omega \vdash N$
Left Inversion	$\Psi; \Delta; \Omega \vdash \bar{P}$

These are the five phases of focusing. In stable sequents, we must choose a proposition to focus on. During the focus phases, we must continue to decompose the focused proposition, indicated in [brackets]. In the inversion phases, we must eagerly decompose invertible connectives. The rules defining focused linear logic proofs are Figure 2.

3.2 Embedding Linear Logic in FF

To a first approximation, the embedding will work by requiring negative (resp. positive) object language propositions to be associated with a world (resp. frame) to be translated. (We will need to generalize the translation of positives, but this view suffices for now.) That is, we will define a pair of mutually recursive functions $(N @ p)$ and $(P @ f)$, both of which yield a negative FF proposition A. This intentionally resembles our previous use of the expressions $a^- @ f$ or $a^+ @ p$ as positive FF atoms, so that we can define $(\bar{N} @ p)$ and $(\bar{P} @ f)$ by merely abuse of notation.

Before giving the full translation, we wish to further illustrate the way the embedding works with a few small examples. First of all, we describe how certain linear logic judgments are translated. We temporarily ignore unrestricted contexts Ψ for brevity.

A typical stable (that is, having no inversion steps immediately available) focused linear logic sequent such as

$$N_1, \dots, N_n \vdash P \tag{1}$$

will be translated to the FF sequent

$$(N_1 @ \alpha_1), \dots, (N_n @ \alpha_n), (P @ \phi) \vdash_{\text{FF}} \phi \triangleleft (\alpha_1 * (\cdots * \alpha_n)) \tag{2}$$

for $\alpha_1, \dots \alpha_n$ distinct fresh world variables, and ϕ a fresh frame variable. The job before us is to choose how to define the embedding functions so that provability (and proofs) of (2) coincide with that of (1).

An intuition for what is going on in the sequent (2) is that all of the individual pieces of the linear sequent (1) are available in the context of (2), but available as (and this is unavoidable, since FF is not a substructural logic) unrestricted, ordinary hypotheses. However each hypothesis will be effectively 'tagged' with a unique label by the embedding @. The FF atom $\phi \triangleleft (\alpha_1 * (\cdots * \alpha_n))$ that is the conclusion of

$$
\frac{\Psi, N; \Psi; \Delta[N] \vdash \bar{P}}{\Psi, N; \Psi; \Delta \vdash \bar{P}}\,\mathit{copy}
\qquad
\frac{\Psi; \Delta[N] \vdash \bar{P}}{\Psi; \Delta, N \vdash \bar{P}}\,\mathit{foc}
\qquad
\frac{\Psi; \Delta \vdash \bar{P}}{\Psi; \Delta; \cdot \vdash \bar{P}}\,\mathit{blur}
\qquad
\frac{}{\Psi; \Delta, a^{+} \vdash [a^{+}]}\,a^{+}R
\qquad
\frac{\Psi; \Delta, a^{+}; \Omega \vdash N}{\Psi; \Delta; \Omega, a^{+} \vdash N}\,a^{+}L
$$

$$
\frac{}{\Psi; \Delta; [a^{-}] \vdash a^{-}}\,a^{-}L
\qquad
\frac{\Psi; \Delta; \cdot \vdash N}{\Psi; \Delta \vdash [\downarrow N]}\,\downarrow R
\qquad
\frac{\Psi; \Delta, N; \Omega \vdash \bar{P}}{\Psi; \Delta; \Omega, \downarrow N \vdash \bar{P}}\,\downarrow L
\qquad
\frac{\Psi; \Delta; \Omega \vdash P}{\Psi; \Delta; \Omega \vdash \uparrow P}\,\uparrow R
\qquad
\frac{\Psi; \Delta; P' \vdash \bar{P}}{\Psi; \Delta; [\uparrow P'] \vdash \bar{P}}\,\uparrow L
\qquad
\frac{}{\Psi; \Delta; \Omega \vdash \top}\,\top R
$$

$$
\frac{\Psi; \Delta; \Omega, P \vdash N}{\Psi; \Delta; \Omega \vdash P \multimap N}\multimap R
\qquad
\frac{\Psi; \Delta_1 \vdash [P] \quad \Psi; \Delta_2; [N] \vdash \bar{P}}{\Psi; \Delta_1, \Delta_2; [P \multimap N] \vdash \bar{P}}\multimap L
\qquad
\frac{\Psi; \Delta; \Omega \vdash N_1 \quad \Psi; \Delta; \Omega \vdash N_2}{\Psi; \Delta; \Omega \vdash N_1 \mathbin{\&} N_2}\,\&R
\qquad
\frac{\Psi; \Delta; [N_i] \vdash \bar{P}}{\Psi; \Delta; [N_1 \mathbin{\&} N_2] \vdash \bar{P}}\,\&L_i
$$

$$
\frac{\Psi; \Delta \vdash [P_i]}{\Psi; \Delta \vdash [P_1 \oplus P_2]}\oplus R_i
\qquad
\frac{\Psi; \Delta; \Omega, P_1 \vdash N \quad \Psi; \Delta; \Omega, P_2 \vdash N}{\Psi; \Delta; \Omega, P_1 \oplus P_2 \vdash N}\oplus L
\qquad
\frac{}{\Psi; \Delta; \Omega, 0 \vdash N}\,0L
\qquad
\frac{\Psi; \Delta_1 \vdash [P_1] \quad \Psi; \Delta_2 \vdash [P_2]}{\Psi; \Delta_1, \Psi; \Delta_2 \vdash [P_1 \otimes P_2]}\otimes R
$$

$$
\frac{\Psi; \Delta; \Omega, P_1, P_2 \vdash N}{\Psi; \Delta; \Omega, P_1 \otimes P_2 \vdash N}\otimes L
\qquad
\frac{\Psi; \cdot; \cdot \vdash N}{\Psi; \cdot \vdash [!N]}\,!R
\qquad
\frac{\Psi, N; \Delta; \Omega \vdash \bar{P}}{\Psi; \Delta; \Omega, !N \vdash \bar{P}}\,!L
$$

Figure 2. Focused Linear Logic Inference Rules

(2) then serves as the prescription of the linear resource discipline that each hypothesis N_i must be used exactly once, because each 'tag' α_i occurs exactly once in the term.

Note that P ends up on the opposite side of the turnstile after embedding — this is always be the case for positive propositions, as positive object-language hypotheses also are translated to FF conclusions. This is caused by the need to preserve focusing phases, together with the fact that FF is almost entirely made of negative propositions. Therefore positives on the right, (resp. left) where they are synchronous (resp. asynchronous), must be translated to negatives on the left (resp. right) where *they* are synchronous (resp. asynchronous).

An entirely negative right-inversion judgment with an empty asynchronous context

$$N_1, \ldots, N_n; \cdot \vdash N \tag{1a}$$

will be translated to the FF sequent

$$(N_1 @ \alpha_1), \ldots, (N_n @ \alpha_n) \vdash_{\text{FF}} N @ (\alpha_1 * (\cdots * \alpha_n)) \tag{2a}$$

Again, it is a world expression $(\alpha_1 * (\cdots * \alpha_n))$ that prescribes the *linear* context in which N must be proved.

With these we can consider the negative linear proposition $\downarrow N_1 \multimap N_2$. Its translation $(\downarrow N_1 \multimap N_2)@p$ at world p will turn out to be

$$\forall \alpha . \downarrow (N_1 @ \alpha) \to (N_2 @ (p * \alpha)) \tag{$*$}$$

To prove $(*)$ in FF, we add a new hypothesis $(N_1 @ \alpha)$ to the context, for a fresh α, and try to prove $(N_2 @ (p * \alpha))$. This new goal is still of the form in (2a), and corresponds to a linear context with one more hypothesis in it.

If we were to focus on $(*)$ as a hypothesis in FF it would require us to *choose* a world expression q to substitute for α, to prove as a new goal $(N_1 @ q)$, and to continue on proving our original goal with the new hypothesis $(N_2 @ (p * q))$. Here we fall outside our simplified version of the invariant maintained by the embedding (for $(p * q)$ is no longer generally a world variable) but we can at least point out that the choice of q corresponds to the *choice of resources* Δ_1 to devote to proving P in the $\multimap L$ rule. By making the world that N_2 is translated at the larger expression $p * q$, we are indicating that *more resources* are required to actually obtain N_2: we must spend the p that were already associated with obtaining $\downarrow N_1 \multimap N_2$, plus the additional q that were used to produce N_1.

Consider also the (negative, because of $\uparrow$) proposition $\uparrow (a_1^{+} \otimes a_2^{+})$. Imagine that it occurs as a hypothesis within a sequent like (2), say at world α, as $\uparrow (a_1^{+} \otimes a_2^{+}) @ \alpha$. This will be translated as

$$\forall \phi . \uparrow (\forall \alpha_1 . (a_1^{+} @ \alpha_1) \to \forall \alpha_2 . (a_2^{+} @ \alpha_2) \to \phi \lhd \alpha_1 * \alpha_2) \to \phi \lhd \alpha \tag{$**$}$$

If we focus on this proposition on the left in FF, it makes us choose a frame f to substitute for ϕ — and then, when focus finally reaches the atom $f \lhd \alpha$ (what once was $\phi \lhd \alpha$) it must match (up to equivalence) the current conclusion of (2), which represents the current shape of the linear logic sequent. If that succeeds, we then proceed by trying to prove $\forall \alpha_1 . (a_1^{+} @ \alpha_1) \to \forall \alpha_2 . (a_2^{+} @ \alpha_2) \to \phi \lhd \alpha_1 * \alpha_2$, which adds two new atoms to the context (each at a fresh world variable) and updates the current sequent-structure expression to contain them.

Note how critical focusing discipline is in this case to reasoning about the behavior of FF proofs: if not for inversion and focus we would have had to consider many more different possible proofs of $(**)$ that paused and interleaved decomposition of it with other parts of the sequent. Showing that the embedding faithfully reproduced provability of linear logic would be much harder, and the claim that it faithfully reproduced proofs would be false!

3.2.1 Embedding Propositions

We now give the full embedding that gives rise to these particular translations. To do so, we must speak of *world continuations* k, which are defined to be abstractions $\alpha.A$, that is, an FF proposition abstracted over a bound world variable α. We write $k(p)$ for application of k to an argument p. We define $(\alpha.A)(p)$ to be the substitution $\{p/\alpha\}A$.

The embedding is then two mutually recursive functions linear logic propositions to negative propositions of FF. We translate N with respect to a world p, and write the translation $(N@p)$, and we translate P with respect to a *continuation* k, written $(P@k)$. Inasmuch as proving $(N@p)$ means something like 'proving N given resources p', $(P@k)$ means something like 'refuting P, and then proving what k yields given a representation of the resources that an assumption of P represents'. Because we have structures $f \triangleleft p$, we can canonically construe any frame to also be a continuation. Specifically, we abuse notation and identify a frame f with the continuation $\alpha.(f \triangleleft \alpha)$.

The embedding of propositions is defined in Figure 3. Note that for a positive object-language atom a^+, the expression $(a^+@k)$ is an invocation of the translation function, while $(a^+@p)$ is a positive atom of FF. Conversely, $(a^-@p)$ is a call to translation, but $(a^-@f)$ is an atom. We take advantage of this overloading the definitions below.

3.2.2 Embedding Sequents

We refer to any FF context of the form

$$\Gamma = (\bar{N}_1@\alpha_1), \ldots, (\bar{N}_n@\alpha_n), (\bar{P}_1@\phi_1), \ldots, (\bar{P}_m@\phi_m)$$

as a *regular* context. A regular context represents a collection of object language hypotheses and conclusions, each uniquely labelled by a distinct world or frame variable, respectively. Again, since FF is itself not substructural, the representations of object language hypotheses and (even multiple) conclusions persist in it during bottom-up proof: it is only the world and frame discipline that prevents resources from being inappropriately reused.

Specifically, a world, frame, or structure can be used to select from Γ a collection of substructural hypotheses and/or conclusion, according to the following definition. Suppose Γ is regular, and that Δ is the linear logic context $\bar{N}_{i_1}, \ldots, \bar{N}_{i_\ell}$ for distinct $i_1, \ldots, i_\ell \in \{1, \ldots, n\}$. Then we write:

1. $\Gamma \sim_p \Delta$ iff $p \equiv \alpha_{i_1} * \cdots * \alpha_{i_\ell}$
2. $\Gamma \sim_f (\Delta \vdash P_i)$ iff $f \equiv \phi_i \circledast (\alpha_{i_1} * \cdots * \alpha_{i_\ell})$
3. $\Gamma \sim_{f \triangleleft p} (\Delta \vdash P_i)$ iff $f \triangleleft p \equiv \phi_i \triangleleft (\alpha_{i_1} * \cdots * \alpha_{i_\ell})$

Where one might pronounce $\Gamma \sim_p \Delta$ as 'Γ according to p represents Δ' and so on.

Asynchronous contexts Ω are translated as follows. Define the function $((\Omega; N)@p)$, which yields an FF proposition A from Ω, N, p, and describes inversion of N on the right with Ω on the left, at world p. It is defined by

$$((\Omega, P; N)@p) = (P@\alpha.((\Omega; N)@(p * \alpha)))$$
$$((\cdot; N)@p) = (N@p)$$

The function $(\Omega@f)$ yields a proposition from Ω, f, and describes the inversion of Ω on the left, at frame f. It is defined by

$$((\Omega, P)@f) = (P@\alpha.(\Omega@(f \circledast \alpha))) \qquad (\cdot@f) = f \triangleleft \epsilon$$

Unrestricted contexts Ψ are translated uniformly at ϵ:

$$(\cdot@\epsilon) = \cdot$$
$$((\Psi, N)@\epsilon) = (\Psi@\epsilon), (N@\epsilon)$$

3.3 Adequacy

Recall that to show the embedding is correct, we seek a compositional bijection between proofs before and after translation. This bijection is obtained as the computational content of the theorem that proofs *can* be so translated. There are five parts to the first theorem, corresponding to the five phases of focused proof search: stable sequents, negative focus, positive focus, negative inversion, and positive inversion.

Theorem 3.1 (Adequacy) *Suppose Γ is regular. Let $\Gamma' = (\Psi@\epsilon), \Gamma$.*

1. $\Gamma' \vdash_{FF} s^-$ *iff there are* $\Delta, \bar{P}$ *such that* $\Gamma \sim_s {}^-(\Delta \vdash \bar{P})$ *and* $\Psi; \Delta \vdash \bar{P}$.
2. $\Gamma'; [(N@p)] \vdash_{FF} s^-$ *iff there are* $f, \Delta, \bar{P}$ *such that* $\Gamma \sim_f (\Delta \vdash \bar{P})$ *and* $\Psi; \Delta[N] \vdash \bar{P}$ *and* $f \triangleleft p \equiv s^-$.
3. $\Gamma'; [(P@k)] \vdash_{FF} s^-$ *iff there are* p, Δ *such that* $\Gamma \sim_p \Delta$ *and* $\Psi; \Delta \vdash [P]$ *and* $\Gamma; [k(p)] \vdash_{FF} s^-$.
4. $\Gamma' \vdash_{FF} ((\Omega; N)@p)$ *iff there is* Δ *such that* $\Gamma \sim_p \Delta$ *and* $\Psi; \Delta; \Omega \vdash N$.
5. $\Gamma' \vdash_{FF} (\Omega@f)$ *iff there are* $\Delta, \bar{P}$ *such that* $\Gamma \sim_f (\Delta \vdash \bar{P})$ *and* $\Psi; \Delta; \Omega \vdash \bar{P}$.

Proof Deferred to the appendix.

Theorem 3.2 (Adequacy of Provability) *The mutually recursive functions (for instance the one from derivations of $\Gamma \vdash_{FF} s^-$ and derivations of $\Delta \vdash \bar{P}$ such that $\Gamma \sim_s {}^-(\Delta \vdash \bar{P})$, and so on) given by the constructive proof of the previous theorem is a bijection.*

P	$(P@k)$
0	$\top$
$P_1 \oplus P_2$	$(P_1@k) \wedge (P_2@k)$
1	$k(\epsilon)$
$P_1 \otimes P_2$	$(P_2@\alpha.(P_1@\beta.k(\alpha * \beta)))$
a^+	$\forall\alpha.a^+@\alpha \Rightarrow k(\alpha)$
$\downarrow N$	$\forall\alpha.\downarrow(N@\alpha) \Rightarrow k(\alpha)$
$!N$	$\downarrow(N@\epsilon) \Rightarrow k(\epsilon)$

N	$(N@p)$
$\top$	$\top$
$N_1 \mathbin{\&} N_2$	$(N_1@p) \wedge (N_2@p)$
$P \multimap N$	$(P@\alpha.(N@(p * \alpha)))$
a^-	$\forall\phi.a^-@\phi \Rightarrow \phi \lhd p$
$\uparrow P$	$\forall\phi.\downarrow(P@\phi) \Rightarrow \phi \lhd p$

Figure 3. Linear Logic Embedding

Proof By structural induction on the proof that the previous theorem yields a given result. ■

Corollary 3.3 *Focused linear logic satisfies cut admissibility and identity expansion properties, for example*

$$\frac{\Delta_1; \Omega \vdash N \qquad \Delta_2[N] \vdash \bar{P}}{\Delta_1, \Delta_2; \Omega \vdash \bar{P}}$$

Proof Directly by appeal to cut and identity properties of FF, for we have just shown that focused linear logic is faithfully embedded there. The argument for the cut admissibility property mentioned immediately above amounts to simply cutting (in FF) the translation of N against itself, translated in both positions at a world that represents the combination of Δ_1 and Ω. ■

Theorem 3.4 (Adequacy of Focusing Structure) *The focusing structure of the image of a proof after embedding is isomorphic to to that of the original: synchronous phases correspond to synchronous, and asynchronous to asynchronous.*

Proof By induction over the structure of the derivation. ■

3.3.1 Unpolarized Linear Logic

We can also embed ordinary unpolarized linear logic, and the usual unfocused sequent calculus for it, simply by composing our translation above with the well-known embedding of unpolarized propositions into polarized ones that inserts shift connectives between every connective. The polarization function consists of four mutually recursive pieces, defined in the first two parts of Figure 4. They are $RU, LU, \hat{R}U, \hat{L}U$, where U is an unpolarized linear logic proposition from the grammar

$$\begin{aligned}\text{Propositions } U ::= {} & U \mathbin{\&} U \mid \top \mid U \multimap U \mid a^- \mid \\ & U \otimes U \mid 1 \mid U \oplus U \mid 0 \mid a^+ \mid !U\end{aligned}$$

The functions R and $\hat{L}$ always yield a positive proposition result, and L and $\hat{R}$ always yield a negative.

At a very high level, we can show a theorem such as

Theorem 3.5 *The derivations of* $\vdash_{FF} (\hat{R}U@\epsilon)$ *are in bijective correspondence with the unfocused linear logic proofs of* $\vdash U$.

This can be proved either directly, following the same technique as Theorem 3.1, or by chaining together that theorem with the correctness of the polarization function.

This theorem may seem to be a very special case of what is actually desired, since it only addresses linear sequents with an empty context, but one can easily use implication to build up nontrivial propositions U and see that the theorem suffers no loss of generality.

To show that the unfocused and focused proof systems are equivalent — that is, to show the completeness of focusing for linear logic — we observe that the only difference between a proposition and the unfocused version of it (that is, the result of erasing all shifts in it and repolarizing by the above functions) is merely the creation of some extra double-shifts, and use the following lemma:

Lemma 3.6 $(\uparrow\downarrow N@p) \dashv\vdash (N@p)$ *and* $(\downarrow\uparrow P@k) \dashv\vdash (P@k)$.

This follows by a straightforward induction on N and P, constructing derivations in FF. We claim this is a novel accomplishment of our approach that we are able to derive not only cut elimination theorems but also focusing completeness proofs 'for free' from an algebraic characterization of substructural contexts.

4 Embedding Ordered Logic

In this section we show how to embed a different substructural logic: ordered logic. The study of noncommutative logics goes back much earlier than the history of linear logic, to Lambek [Lam58]. We are also interested in including the modalities ! and ¡, which allow unrestricted and mobile (i.e. satisfying the structural law of exchange) hypotheses to be used alongside ordered hypotheses, as descried by Polakow [Pol01]. Although focused proof search

$$\hat{R}U = \begin{cases} a^- & \text{if } U \text{ is } a^- \\ \uparrow RU & \text{otherwise.} \end{cases}$$

$$\hat{L}U = \begin{cases} a^+ & \text{if } U \text{ is } a^+ \\ \downarrow LU & \text{otherwise.} \end{cases}$$

U	RU	LU
$U_1 \otimes U_2$	$\downarrow\hat{R}U_1 \otimes \downarrow\hat{R}U_2$	$\uparrow(\hat{L}U_1 \otimes \hat{L}U_2)$
$U_1 \oplus U_2$	$\downarrow\hat{R}U_1 \oplus \downarrow\hat{R}U_2$	$\uparrow(\hat{L}U_1 \oplus \hat{L}U_2)$
1	1	$\uparrow 1$
0	0	$\uparrow 0$
$U_1 \& U_2$	$\downarrow(\hat{R}U_1 \& \hat{R}U_2)$	$\uparrow\hat{L}U_1 \& \uparrow\hat{L}U_2$
$\top$	$\downarrow\top$	$\top$
$U_1 \multimap U_2$	$\downarrow(\hat{L}U_1 \multimap \hat{R}U_2)$	$\downarrow\hat{R}U_1 \multimap \uparrow\hat{L}U_2$
a^-	$\downarrow a^-$	a^-
a^+	a^+	$\uparrow a^+$
$!U$	$!\hat{R}U$	$\uparrow!\uparrow\hat{L}U$

U	RU	LU
$U_1 \rightarrowtail U_2$	$\downarrow(\hat{L}U_1 \rightarrowtail \hat{R}U_2)$	$\downarrow\hat{R}U_1 \rightarrowtail \uparrow\hat{L}U_2$
$U_1 \twoheadrightarrow U_2$	$\downarrow(\hat{L}U_1 \twoheadrightarrow \hat{R}U_2)$	$\downarrow\hat{R}U_1 \twoheadrightarrow \uparrow\hat{L}U_2$
$U_1 \bullet U_2$	$\downarrow\hat{R}U_1 \bullet \downarrow\hat{R}U_2$	$\uparrow(\hat{L}U_1 \bullet \hat{L}U_2)$
$¡U$	$¡\hat{R}U$	$\uparrow¡\uparrow\hat{L}U$

Figure 4. Polarization

for ordered logic is not as well-known as for linear logic, it is implicit in, taken together, the ordered logic programming systems of Pfenning and Simmons [PS09] and Polakow [Pol00].

Without its exponentials, the encoding of ordered logic would be almost entirely the same that of linear logic, except that we would drop the axiom that makes $*$ commutative. We wish to illustrate, however, how to simultaneously accommodate the variety of different substructural hypotheses present in full ordered logic.

The propositions of ordered logic are

$$\text{Propositions } U ::= U \,\&\, U \mid \top \mid U \rightarrowtail U \mid U \twoheadrightarrow U \mid a^- \mid U \bullet U \mid 1 \mid U \oplus U \mid 0 \mid a^+ \mid ¡U \mid !U$$

with the noncommutative $\bullet$ replacing linear logic's $\otimes$, and two distinct implications, $\rightarrowtail$ and $\twoheadrightarrow$, which add hypotheses to the left and right end of the context respectively. We can mostly rehearse the polarization translation for linear logic, adding the clauses in the last part of Figure 4, to embed the unpolarized propositions into the syntax of *polarized* ordered logic propositions, defined as follows.

$$\begin{array}{lcl} \text{Negatives } N & ::= & \uparrow P \mid N \,\&\, N \mid \top \mid P \rightarrowtail N \mid P \twoheadrightarrow N \mid a^- \\ \text{Positives } P & ::= & \downarrow N \mid P \bullet P \mid 1 \mid P \oplus P \mid 0 \mid a^+ \mid !N \mid ¡N \end{array}$$

Here we did have to make certain choices about which propositions to try to make negative or positive. Although these choices were not automatic, they were extremely natural to make by analogy with focusing intuitions in linear logic: we already expect multiplicative conjunctions to be positive, and implications to be negative, with positive arguments.

The FF embedding of polarized ordered logic works as follows. We have four sorts,

$$\text{Sorts } \sigma ::= \mathsf{world} \mid \mathsf{linworld} \mid \mathsf{frame} \mid \mathsf{struct}$$

Where we now have two separate notions of (ordered) worlds and mobile linear worlds, to accommodate the inclusion of $¡$, which allows mixture of ordered and mobile hypotheses. We add function symbols sufficient to make the syntax of these sorts effectively the following:

$$\begin{array}{rcl} \text{Linear Worlds } \tilde{p} & ::= & \tilde{\alpha} \mid \tilde{\epsilon} \mid \tilde{p} * \tilde{p} \\ \text{Worlds } p & ::= & \alpha \mid \epsilon \mid p \cdot p \mid \iota\tilde{p} \\ \text{Frames } f & ::= & \phi \mid f \odot p \mid p \odot f \end{array}$$

Here ι is an *inclusion* from linear worlds into ordered worlds. We axiomatize $\equiv$ by

$$\epsilon \cdot p \equiv p \qquad p \cdot (q \cdot r) \equiv (p \cdot q) \cdot r$$

$$(f \odot p) \triangleleft q \equiv f \triangleleft (q \cdot p) \qquad (p \odot f) \triangleleft q \equiv f \triangleleft (p \cdot q)$$

$$\tilde{\epsilon} * \tilde{p} \equiv \tilde{p} \qquad \tilde{p} * \tilde{q} \equiv \tilde{q} * \tilde{p} \qquad \tilde{p} * (\tilde{q} * \tilde{r}) \equiv (\tilde{p} * \tilde{q}) * \tilde{r}$$

$$\iota\tilde{\epsilon} \equiv \epsilon \qquad \iota(\tilde{p} * \tilde{q}) \equiv \iota\tilde{p} \cdot \iota\tilde{q}$$

which makes $*$ commutative, $\cdot$ generally noncommutative, (but commutative on the range of ι) and which makes ι an algebra homomorphism. What we have done is axiomatized a noncommutative monoid with a commutative submonoid: the latter describes exactly the mobile hypotheses. The language of frames has expanded, as noted above, because the presence of ordered hypotheses means that a context with a hole can be built up by adjoining hypotheses to the left or right of that hole.

The translation of propositions for ordered logic is in Figure 5. In the absence of a definitive focusing calculus for ordered logic, the main result is similar to Theorem 3.5.

Theorem 4.1 *The derivations of* $\vdash_{FF} (\hat{R}U @ \epsilon)$ *are in bijective correspondence with the unfocused ordered logic proofs of* $\vdash U$.

The same comment as for Theorem 3.5 about the generality of the result applies here.

Furthermore, since we staging the embedding in two parts — polarization followed by the embedding of polarized ordered propositions — we can observe that the latter amounts to specifying a focusing proof system for ordered logic. We could in fact quite straightforwardly work backwards from the FF proofs that are possible on embedded

P	$(P@k)$
0	$\top$
$P_1 \oplus P_2$	$(P_1@k) \wedge (P_2@k)$
1	$k(\epsilon)$
$P_1 \bullet P_2$	$(P_1@\alpha.(P_2@\beta.k(\alpha \cdot \beta)))$
a^+	$\forall\alpha.a^+@\alpha \Rightarrow k(\alpha)$
$\downarrow N$	$\forall\alpha.\downarrow(N@\alpha) \Rightarrow k(\alpha)$
$¡N$	$\forall\tilde{\alpha}.(N@\iota\tilde{\alpha}) \Rightarrow k(\iota\tilde{\alpha})$
$!N$	$(N@\epsilon) \Rightarrow k(\epsilon)$

N	$(N@p)$
$\top$	$\top$
$N_1 \,\&\, N_2$	$(N_1@p) \wedge (N_2@p)$
$P \rightarrowtail N$	$(P@\alpha.(N@(\alpha \cdot p)))$
$P \twoheadrightarrow N$	$(P@\alpha.(N@(p \cdot \alpha)))$
a^-	$\forall\phi.a^-@\phi \Rightarrow \phi \triangleleft p$
$\uparrow P$	$\forall\phi.\downarrow(P@\phi) \Rightarrow \phi \triangleleft p$

Figure 5. Ordered Logic Embedding

propositions in order to invent a collection of first-class inference rules that mimics it exactly, generalizing the calculus in [Pol01]; only space limitations prevent us from doing this here. The important thing is that an exact analogue of Lemma 3.6 still holds, meaning that the focused version of ordered logic, regardless of how it is implemented, is complete with respect to unfocused proof.

5 Related Work

Our present approach generalizes prior work [Ree07] that can be seen as achieving roughly the same goals for just the negative fragment of linear logic. The key insight that allowed us to embed the entire logic was the consideration of *frames* as well as worlds — only the latter were considered in [Ree07].

There has been a significant amount of work on the Kripke semantics for substructural logics, including the modalities—see Kamide [Kam02] for a systematic study and further references. Similarly, the logic of bunched implication was conceived from the beginning with a resource semantics [OP99]. The classical nature of the metalanguage in which these interpretations are formulated becomes particularly apparent when the objective of the translation is theorem proving: proof search then proceeds in a classical logic where formulas have been augmented in a systematic way to encompass worlds, be it for linear [MO99], noncommutative [GN03], or bunched [GMP05] logic. At the root of these concrete interpretations we can find labeled deduction [BDG$^+$00]. Our work shares with these the idea of resource combination via algebraic operators and partial orders for resource entailment.

The above models capture a notion of truth with respect to resources or worlds. In many applications, however, we are interested in the precise structure of proofs. Besides well-known computational interpretations of proofs, their fine structure also determines the behavior of logic programs where computation proceeds by proof search. Capturing proofs is usually the domain of logical frameworks such as LF [HHP93], or its substructural extensions such as RLF [IP98], LLF [CP02], or OLF [Pol01], where proofs are reified as objects. In case of these frameworks, however, natural representations (namely those mapping the substructural consequence of the object logic to consequence in the metalogic) are limited by the substructural properties of the framework. Moreover, even if (small-step) proofs can be represented in this manner, focused proofs in an object language seem to require even more substructural expressiveness in the metalanguage. Instead of escalating the number of substructural judgments and modalities in the metalanguage, we propose here to slice through the knot using just an intuitionistic framework and capture substructural properties algebraically. The framework here is first-order, but we conjecture, based on our experience with in HLF [Ree07], that reifying proofs in a dependent version of the present proposal should not present much difficulty.

6 Conclusion

We have presented a method of embedding substructural logics into a first-order focused constructive logic which preserves proofs and focusing structure. It isolates the algebra of contexts from the machinery of propositional inference, and so allows further comparison between otherwise superficially different logics — although intuitively it is quite clear that linear logic's $\otimes$ and ordered logic's $\bullet$ internalize an operation on contexts, we can give this intuition formal content by saying precisely that they share the same translation up to a different choice of algebra. In future work, we hope to extend this semantics to classical variants of substructural logics, as well as modal logics such as judgmental S4 [PD01], by analogy between $\Box$ and $!$. It also seems like it might be possible to unify linear with bunched logic along the same lines as ordered and linear logic, by using a modality to mediate between the two respective algebraic structures.

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A Appendix

A.1 Proof of Identity Expansion

We strengthen the induction hypothesis to:
For all Γ, A,

1. If there is a derivation

$$\begin{array}{c} \Gamma; [A] \vdash s^- \\ \vdots \\ \Gamma \vdash s^- \end{array}$$

   parametric in s^-, then $\Gamma \vdash A$.

2. $\Gamma, B \vdash [B]$

The proof of it is by induction on A, with case 1 considered less than case 2 for equal A.

1. Split cases on A. We show some representative cases.

Case: s^-. Use reflexivity of $\equiv$.

$$\frac{s^- \equiv s^-}{\Gamma; [s^-] \vdash s^-}$$

Case: $A_1 \mathbin{\&} A_2$. Apply the i.h. to

$$\begin{array}{c} \dfrac{\Gamma; [A_i] \vdash s^-}{\Gamma; [A_1 \mathbin{\&} A_2] \vdash s^-} \\ \vdots \\ \Gamma \vdash s^- \end{array}$$

to get $\Gamma \vdash A_i$, and then observe

$$\frac{\Gamma \vdash A_1 \qquad \Gamma \vdash A_2}{\Gamma \vdash A_1 \mathbin{\&} A_2}$$

Case: $B \Rightarrow A$. Apply the i.h. at A to

$$\begin{array}{c} \dfrac{\dfrac{}{\Gamma, B \vdash [B]}\,i.h. \qquad \Gamma, B; [A] \vdash s^-}{\Gamma, B; [B \Rightarrow A] \vdash s^-} \\ \vdots \\ \Gamma, B \vdash s^- \end{array}$$

(having used the i.h. part 2 at B to see that B entails itself) to obtain $\Gamma, B \vdash A$, and then prove

$$\frac{\Gamma, B \vdash A}{\Gamma \vdash B \Rightarrow A}$$

2. If B we are done. Otherwise, $B = \downarrow A$. In that case, blur on the right, and apply i.h. part 1 to the inference rule $\downarrow$ L.

■

A.2 Proof of Adequacy

danger, danger, obsolete notation By lexicographic induction on the object-language proposition (or in parts 4,5, context), and the derivation. Some representative cases:

1. In the forward direction, the only move available is focusing on some proposition in Γ. Since by assumption Γ is regular, it is either of the form $(\bar{N}@\alpha)$ or $(\bar{P}@\phi)$. But we cannot begin focus on a positive atom on the left, so the only possibilities left are $(N@\alpha)$ or $(P@\phi)$. For these apply the induction hypothesis part 2 or 3, respectively.

   In the reverse direction, we likewise appeal to induction hypothesis 2 or 3 depending on whether a negative or positive atom is focused on.

2.

Case: $P \multimap N$ in the forward direction. By assumption $\Gamma; [(P \multimap N@p)] \vdash_{\text{FF}} s^-$ which means $\Gamma; [(P@\alpha.(N@(p * \alpha)))] \vdash_{\text{FF}} s^-$. By induction hypothesis part 3, there are q, Δ_1 such that $\Gamma \sim_q \Delta_1$ and $\Delta_1 \vdash [P]$ and $\Gamma; [(N@(p * q))] \vdash_{\text{FF}} s^-$. By the induction hypothesis part 2, there are $f, \Delta_2, \bar{P}$ such that $\Gamma \sim_f (\Delta_2 \vdash \bar{P})$ and $\Delta_2 \vdash [N] > \bar{P}$ and $f \lhd (p * q) \equiv s^-$. To satisfy our obligations we produce the frame $f \circledast q$, the combined context (Δ_1, Δ_2) and conclusion $\bar{P}$, the fact that $\Gamma \sim_{f \circledast q} (\Delta_1, \Delta_2 \vdash \bar{P})$ by definition of $\sim$, the fact that $\Delta_1, \Delta_2 \vdash [P \multimap N] > \bar{P}$ by rule application, and that $(f \circledast q) \lhd p \equiv f \lhd (p * q) \equiv s^-$.

Case: $\uparrow P$ in the forward direction. We have

$$\Gamma; [\forall \phi. \downarrow (P@\alpha.(\phi \lhd \alpha)) \Rightarrow \phi \lhd p] \vdash_{\text{FF}} s^-$$

By inversion, a proof of this chooses f to instantiate ϕ, and contains subderivations of

$$\Gamma \vdash (P@\alpha.(f \lhd \alpha))$$

$$\Gamma; [f \lhd p] \vdash s^-$$

The second of these will only succeed of $f \lhd p \equiv s^-$. We have satisfied part of our obligations by producing the frame f, and this equivalence. For the rest, we appeal to the induction hypothesis part 5, observing that $(P@\alpha.(f \lhd \alpha))$ and $(P@f)$ (the latter being the translation of asynchronous context that just happens to have one element) are the same proposition up to $\equiv$.

4.

Case: $\Omega, (P_1 \otimes P_2)$. We have $\Gamma \vdash_{\text{FF}} ((\Omega, (P_1 \otimes P_2); N)@p)$, whose conclusion by definition is

$$= ((P_1 \otimes P_2)@\alpha.((\Omega; N)@(p * \alpha)))$$

$$= (P_2 @ \alpha_2.(P_1 @ \alpha_1.((\Omega; N) @ (p * (\alpha_1 * \alpha_2)))))$$
$$\equiv (P_2 @ \alpha_2.(P_1 @ \alpha_1.((\Omega; N) @ ((p * \alpha_2) * \alpha_1)))))$$
$$= (P_2 @ \alpha_2.((\Omega, P_1; N) @ (p * \alpha_2)))$$
$$= ((\Omega, P_1, P_2; N) @ p)$$

so we may appeal to the induction hypothesis to find Δ such that $\Gamma \sim_p \Delta$ and $\Delta \vdash \Omega, P_1, P_2 > N$, and use rule application to get $\Delta \vdash \Omega, (P_1 \otimes P_2) > N$.

Case: $\Omega = \cdot$ and $N = \uparrow P$. We know $\Gamma \vdash \forall\phi.\downarrow(P @ \alpha.(\phi \triangleleft \alpha)) \Rightarrow \phi \triangleleft p$. By inversion

$$\Gamma, \downarrow(P @ \alpha.(\phi \triangleleft \alpha)) \vdash \phi \triangleleft p$$

So we are able to appeal to the induction hypothesis part 1.

■

A.3 Polarizing Linear Logic

polar???

These have the property that

Lemma A.1 *Focused linear logic proofs of $LU_1, LU_2, \cdots LU_n \vdash RU$ are in bijective correspondence with unfocused proofs of $U_1, U_2, \cdots U_n \vdash U$*

Proof By induction on the respective derivations. ■

Since we have used a translation that translates propositions differently depending on whether they essentially appear on the left or right, cut elimination and identity no longer come entirely for free. Nonetheless they are not difficult to show; they follow from the following results concerning shift connectives.

Lemma A.2 $(\uparrow\downarrow\uparrow P @ p) \dashv\vdash (\uparrow P @ p)$ *and* $(\downarrow\uparrow\downarrow N @ k) \dashv\vdash (\downarrow N @ k)$.

Lemma A.3 $(\uparrow RU @ p) \dashv\vdash (LU @ p)$ *and* $(RU @ k) \dashv\vdash (\downarrow LU @ k)$.

The first of these two results is not unlike Brouwer's theorem $\neg\neg\neg A \dashv\vdash \neg A$. The second follows from applying it inductively at every connective, and shows that the translations L and R only differ up to the insertion of shift connectives. As a consequence we recover identity expansion and cut admissibility hold for the object language. For example,

Lemma A.4 $U \vdash U$

Proof This sequent polarizes to $LU \vdash RU$, which under translation corresponds to $(LU @ \alpha), (RU @ \phi) \vdash \phi \triangleleft \alpha$. But this is provable iff $(LU @ \alpha) \vdash \forall\phi.(RU @ \phi) \Rightarrow \phi \triangleleft \alpha = (\uparrow RU @ \alpha)$ is by inversion, which we know from Lemma A.3. ■

The cut admissibility proof works similarly: the import of Lemma A.3 is that, while shifts were inserted in slightly different places to exactly match unfocused proof search, they have no effect on provability.