A Comonadic Generalization of **Top**

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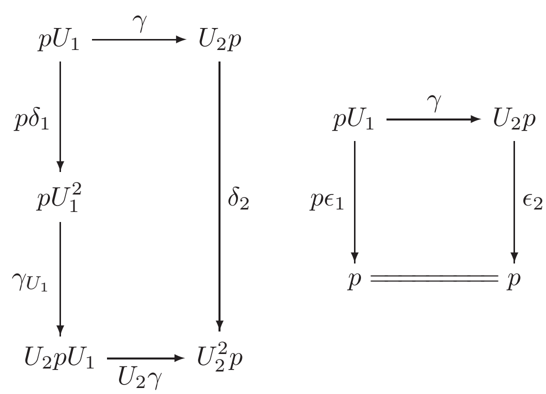
December 6, 2006

Consider an object $P : \mathbf{C} \to \mathbf{Cat}$ of the coslice category $\mathbf{CAT}/\mathbf{Cat}$. A *$P$-space* is defined as pair (C, U, ϵ, δ) where C is an object of $\mathbf{C}$, and U is a comonad (with counit ϵ and comultiplication δ) in the category PC. We usually just refer to C when the naming of the remaining pieces is evident. A *P-continuous map* $C_1 \to C_2$ between P-spaces is given by a pair (f, γ) where $f : C_1 \to C_2$ and γ is a natural transformation $Pf \circ U_1 \to U_2 \circ Pf$ such that (abbreviating $Pf = p$)



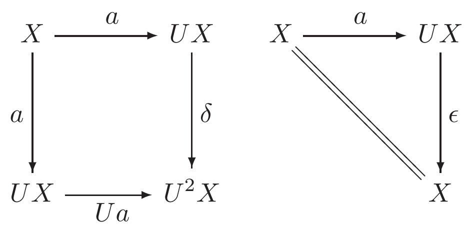
In other words, γ is a coalgebra morphism $\gamma_{U_1} \circ (p\delta_1) \to \delta_2$, and also $p\epsilon_1 \to \epsilon_2$, acting on coalgebras for the functor U_2, and the constantly-p functor, respectively.

Composition and identities are defined by

$$(f', \gamma') \circ (f, \gamma) = (f' \circ f, (\gamma' * Pf) \circ (Pf' * \gamma))$$

$$id_{C,U} = (id_C, id_U)$$

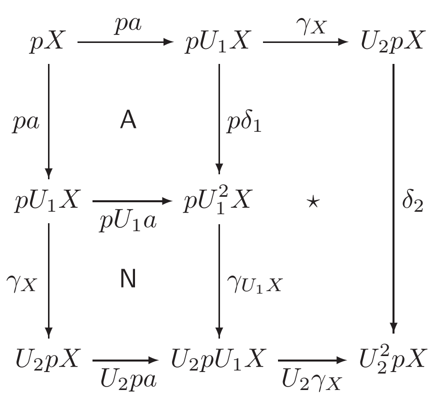
Thus we get a category $P\mathbf{Spa}$ of P-spaces and P-continuous maps. An *open object* of a P-space C is a U-coalgebra, an arrow $a : X \to UX$ in PC satisfying 'comonoid action' axioms with respect to the comonad:

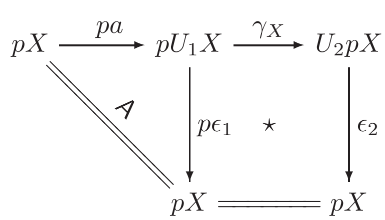


Lemma 0.1 *There is a functor $Op : P\mathbf{Spa} \to \mathbf{Cat}$, which takes a P-space and yields the category its of open objects.*

Proof Arrows in $Op(C)$ are the standard notion of coalgebra morphism. The effect of Op on an arrow in $P\mathbf{Spa}$ is as follows. We take in (f, γ) a P-continuous map $C_1 \to C_2$, and must output a functor $Op(C_1) \to Op(C_2)$. First we define the object part of this functor: if $a : X \to UX$ is an open object in C_1, then we claim $\gamma_X \circ Pf(a)$ is an open object in C_2, with underlying object $Pf(X)$.

We must check that the comonad algebra axioms hold. Abbreviate again $Pf = p$. Cells marked A follow by hitting assumptions with p, N follows by naturality of γ, and $\star$ are from the definition of P-continuous.





For the arrow part of the functor $Op(C_1) \to Op(C_2)$ we must consider composition of coalgebra morphisms, but these are preserved by $a \mapsto \gamma_X \circ pa$:

$$\begin{array}{ccccc}
pX & \xrightarrow{pa} & pU_1X & \xrightarrow{\gamma_X} & U_2pX \\
{\scriptstyle pf}\downarrow & \mathsf{A} & \downarrow{\scriptstyle pU_1f} & \mathsf{N} & \downarrow{\scriptstyle U_2pf} \\
pY & \xrightarrow[pb]{} & pU_1Y & \xrightarrow[\gamma_Y]{} & U_2pY \\
{\scriptstyle pg}\downarrow & \mathsf{A} & \downarrow{\scriptstyle pU_1g} & \mathsf{N} & \downarrow{\scriptstyle U_2pg} \\
pZ & \xrightarrow[pc]{} & pU_1Z & \xrightarrow[\gamma_Z]{} & U_2pZ
\end{array}$$

as are identities:

$$\begin{array}{ccccc}
pX & \xrightarrow{pa} & pU_1X & \xrightarrow{\gamma_X} & U_2pX \\
{\scriptstyle p(id)}\downarrow & \mathsf{A} & \downarrow{\scriptstyle pU_1(id)} & \mathsf{N} & \downarrow{\scriptstyle U_2p(id)} \\
pX & \xrightarrow[pa]{} & pU_1X & \xrightarrow[\gamma_X]{} & U_2pX
\end{array}$$

■

Theorem 0.2 *Let P be the functor $\mathbf{Sets}^{\mathrm{op}} \to \mathbf{Cat}$ that takes X to the evident poset category arising from its powerset $\mathcal{P}X$ ordered by inclusion, and takes a function $f : X \to Y$ to its inverse image map $f^{<} : \mathcal{P}Y \to \mathcal{P}X$. Then the category $\mathbf{Top}$ is isomorphic to $P\mathbf{Spa}^{\mathrm{op}}$.*

Proof For a P-space C, take C as the underlying set of the topology, and the open objects of C as the open sets of the topology. Given a topological space $(X, \mathcal{T})$, let U be the interior operation. The comonad data ϵ and δ simply record the decreasing and idempotent properties of the interior operation in a topological space. Then (X, U, ϵ, δ) is a P-space. Check that the two definitions of continuity match up. ■

Conjecture There is a nice class of maps from the open objects of C_1 to the open objects of C_2 such that every map that belongs to this class arises from a P-continuous map.